

# DECAY OF CORRELATIONS FOR NORMALLY HYPERBOLIC TRAPPING

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**ABSTRACT.** We prove that for evolution problems with normally hyperbolic trapping in phase space, correlations decay exponentially in time. Normal hyperbolic trapping means that the trapped set is smooth and symplectic and that the flow is hyperbolic in directions transversal to it. Flows with this structure include contact Anosov flows [21],[37], classical flows in molecular dynamics [25],[26], and null geodesic flows for black holes metrics [17],[18],[42]. The decay of correlations is a consequence of the existence of resonance free strips for Green's functions (cut-off resolvents) and polynomial bounds on the growth of those functions in the semiclassical parameter.

## 1. STATEMENT OF RESULTS

**1.1. Introduction.** We prove the existence of resonance free strips for general semiclassical problems with normally hyperbolic trapped sets. The width of the strip is related to certain Lyapunov exponents and, for the spectral parameter in that strip, the Green's function (cut-off resolvent) is polynomially bounded. Such estimates are closely related to exponential decay of correlations in classical dynamics and in scattering problems. The framework to which our result applies covers both settings.

To illustrate the results in a simple setting consider

$$(1.1) \quad P = -h^2 \Delta + V(x), \quad V \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}).$$

The classical flow  $\varphi_t : (x(0), \xi(0)) \mapsto (x(t), \xi(t))$  is obtained by solving Newton's equations  $x'(t) = 2\xi(t)$ ,  $\xi'(t) = -\nabla V(x(t))$ . The trapped at energy  $E$ ,  $K_E$ , is defined as the set of  $(x, \xi)$  such that  $p(x, \xi) \stackrel{\text{def}}{=} \xi^2 + V(x) = E$  and  $\varphi_t(x, \xi) \not\rightarrow \infty$ ,  $t \rightarrow \pm\infty$ .

The flow  $\varphi_t$  is said to be *normally hyperbolic near energy  $E$* , if for some  $\delta > 0$ ,

$$(1.2) \quad K^\delta \stackrel{\text{def}}{=} \bigcup_{|E-E'| < \delta} K_{E'} \text{ is a smooth symplectic manifold, and}$$

the flow  $\varphi_t$  is hyperbolic in the directions transversal to  $K^\delta$ ,

see (1.17) below for a precise definition, and [26] for physical motivation for considering such dynamical setting. A simplest consequence of Theorems 2 and 6 is the following result about decay of correlations.

**Theorem 1.** *Suppose that  $P$  is given by (1.1) and that (1.2) holds, that is the classical flow is normally hyperbolic near energy  $E$ . Then for  $\psi \in \mathcal{C}_c^\infty((E - \delta/2, E + \delta/2))$ , and any  $f, g \in L^2(\mathbb{R}^n)$ , with  $\|f\|_{L^2} = \|g\|_{L^2} = 1$ ,  $\text{supp } f, \text{supp } g \subset B(0, R)$ ,*

$$(1.3) \quad \left| \langle e^{-itP/h} \psi(P)f, g \rangle_{L^2(\mathbb{R}^n)} \right| \leq \frac{C_R \log(1/h)}{h^{1+\gamma c_0}} e^{-\gamma t} + C_{R,N} h^N, \quad t > 0,$$

for any  $\gamma < \lambda_0$  and for all  $N$ . Here  $\lambda_0$  and  $c_0$  are the same as in (1.18) and  $C_R, C_{R,N}$  are constants depending on  $R$  and on  $R$  and  $N$ , respectively.

This means that the correlations decay rapidly in the semiclassical limit: we start with a state localized in space (the support condition) and energy,  $\psi(P)f$ , propagate it, and test it against another spatially localized state  $g$ . The estimate (1.3) is a consequence of the existence of a band without scattering resonances and estimates on cut-off resolvent given in Theorem 2. When there is no trapping, that is when  $K_E = \emptyset$ , then the right hand side in (1.3) can be replaced by  $\mathcal{O}((h/t)^\infty)$ , provided that  $t > T_E$ , for some  $T_E$  – see for instance [30, Lemma 4.2]. On the other hand when strong trapping is present, for instance when the potential has an interaction region separated from infinity by a barrier, then the correlation does not decay – see [30] and references given there.

More interesting quantitative results can be obtained for the wave equation or for decay of classical correlations: see §1.2 for motivation and [42, Theorem 3] and Theorem 5 below for examples. When the outgoing and incoming sets at energy  $E$ ,

$$\Gamma_E^\pm \stackrel{\text{def}}{=} \{(x, \xi) : p(x, \xi) = E, \varphi_t(x, \xi) \not\rightarrow \infty, t \rightarrow \mp\infty\},$$

are sufficiently regular, Theorem 1 and Theorem 2 below (without the specific constant  $\lambda_0$ ) are already a consequence of earlier work by Wunsch-Zworski [42, Theorem 2], and, in the case of closed trajectories, Christianson [9],[10]. For a survey of other recent results on resolvent estimates in the presence of weak trapping we refer to [40].

When normal hyperbolicity is strengthened to  $r$ -normal hyperbolicity for large  $r$  (which implies that  $\Gamma_E^\pm$  are  $C^r$  manifolds) and provided a certain pinching condition on Lyapunov exponents is satisfied, much stronger results have been obtained by Dyatlov [19]. In particular, [19] provides an asymptotic counting law for scattering resonances *below* the band without resonances given in Theorems 2, 4 and 6. It shows the optimality of the size of the band in a large range of settings, for instance, for perturbations of Kerr-de Sitter black holes.

Similar results on asymptotic counting laws in strips have been proved by Faure-Tsujii in the case of Anosov diffeomorphisms [22], and recently announced in the case of contact Anosov flows [23]. In the latter situation, described in Theorem 4 below, the trapped set is a normally hyperbolic smooth symplectic manifold, but the dependence of the stable and unstable subspaces on points on the trapped set is typically highly nonsmooth.

In this paper we do *not* assume any regularity on  $\Gamma_E^\pm$  and provide a quantitative estimate on the resonance free strip. For operators with analytic coefficients this result was already

obtained by Gérard-Sjöstrand [25] with even weaker assumptions on  $K^\delta$ . A new component here, aside from dropping the analyticity assumption, is the bound on the Green's function/resolvent that allows applications to the decay of correlations.

The proof is given first for an operator with a complex absorbing potential. This allows very general assumptions which can then be specialized to scattering and dynamical applications.

Finally we comment on the comparison between the resonance free regions in this paper and the results of [32] where an existence of a pole free strip was given for *hyperbolic* trapped sets, provided a certain *pressure condition* was satisfied. In the setting of [32] the trapped set is typically very irregular but, the assumptions of [32] also include the situation where  $K^\delta$  is a smooth symplectic submanifold, and the flow is hyperbolic both transversely to  $K^\delta$  and on each  $K_E$ . In that case the resonance gap obtained in [32] involves a topological pressure associated with the full (that is, longitudinal and transverse) unstable Jacobian, namely

$$(1.4) \quad \mathcal{P}\left(-\frac{1}{2}(\log J_{\parallel}^+ + \log J_{\perp}^+)\right) = \sup_{\mu} \left( H(\mu) - \frac{1}{2} \int (\log J_{\parallel}^+ + \log J_{\perp}^+) d\mu \right),$$

where the supremum is taken over all flow-invariant probability measures on  $K^\delta$  and  $H(\mu)$  is the Kolmogorov-Sinai entropy of the measure  $\mu$  with respect to the flow. The bound is nontrivial only if this pressure is negative. In the case of mixing Anosov flows discussed in §9 the transverse and longitudinal unstable Jacobians are equal to each other; the above pressure is then equal to the pressure  $\mathcal{P}(-\log J_{\parallel}^+)$ , equivalent with the pressure  $\mathcal{P}(-\log J^u)$  of the Anosov flow, which is known to vanish [6, Proposition 4.4], and hence gives only a trivial bound. For this situation, our spectral bound (Theorem 4) is thus sharper than the pressure bound. On the other hand, one can construct examples where the longitudinal and transverse unstable Jacobians are independent of one another, and such that the pressure (1.4) is more negative — hence sharper — than the value  $-\lambda_0$  given in (1.19), which may be expressed as  $-\lambda_0 = \sup_{\mu} \left( -\frac{1}{2} \int \log J_{\perp}^+ d\mu \right)$ .

**Notation.** We use the following notation  $g = \mathcal{O}_k(f)_V$  means that  $\|g\|_V \leq C_k f$  where the norm (or any seminorm) is in the space  $V$ , and the  $C_k$  depends on  $k$ . When either  $k$  or  $V$  are dropped then the constant is universal or the estimate is scalar, respectively. When  $F = \mathcal{O}_k(f)_{V \rightarrow W}$  then the operator  $F : V \rightarrow W$  has its norm bounded by  $C_k f$ .

**1.2. Motivation.** To motivate the problem we consider the following elementary example. Let  $X = \mathbb{R}$  and  $P = -\partial_x^2$ . A wave evolution is given by  $U(t) \stackrel{\text{def}}{=} \sin(\sqrt{P}t)/\sqrt{P}$ . Then for  $f, g \in \mathcal{C}_c^\infty(\mathbb{R})$  and any time  $t \in \mathbb{R}$  we define the wave *correlation function* as

$$(1.5) \quad C(f, g)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} [U(t)f](x) g(x) dx$$

In this 1-dimensional setting, the correlation function becomes very simple for large times. Indeed, for a certain  $T > 0$  depending on the support of  $f$  and  $g$ , it satisfies

$$\forall t \geq T, \quad C(f, g)(t) = \frac{1}{2} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(x) dx$$

This particular behaviour is due to the fact that the resolvent of  $P$ ,

$$R(\lambda) \stackrel{\text{def}}{=} (P - \lambda^2)^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad \text{Im } \lambda > 0,$$

continues meromorphically to  $\mathbb{C}$  in  $\lambda$  as an operator  $L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}$  and has a pole at  $\lambda = 0$ . In this basic case we see this from an explicit formula,

$$[R(\lambda)f](x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y) dy.$$

More generally, we can consider  $P = -\partial_x^2 + V(x)$ ,  $V \in L_c^\infty(\mathbb{R})$ , with  $V \geq 0$ , for simplicity. With the same definition of  $U(t)$  we now have the Lax-Phillips expansion generalizing (1.5)

$$(1.6) \quad C(f, g)(t) = \int_{\mathbb{R}} U(t)f g dx = \sum_{\text{Im } \lambda_j > -A} e^{-i\lambda_j t} \int_{\mathbb{R}} f u_j dx \int_{\mathbb{R}} g u_j dx + \mathcal{O}(e^{-At}),$$

where  $\lambda_j$  are the poles of the meromorphic continuation of  $R(\lambda) = (P - \lambda^2)^{-1}$  (for simplicity assumed to be simple), and  $u_j$  are solutions to  $(P - \lambda_j^2)u_j = 0$  satisfying  $u_j(x) = a_{\text{sgn } x} e^{i\lambda_j |x|}$  for  $|x| \gg 1$ . Since  $u_j$  are not in  $L^2$  their normalization is a bit subtle: they appear in the residues of  $R(\lambda)$  at  $\lambda_j$ .

The expansion (1.6) makes sense since the number of poles of  $R(\lambda)$  with  $\text{Im } \lambda > -A$  is finite for any  $A$ . If we define  $C(f, g)$  to be 0 for  $t \leq 0$ , (1.6) gives (provided 0 is not a pole of  $R(\lambda)$ ),

$$(1.7) \quad \widehat{C(f, g)}(-\lambda) = \sum_{\text{Im } \lambda_j > -A} \frac{c_j}{\lambda_j - \lambda} + \mathcal{O}\left(\frac{1}{A}\right), \quad c_j \stackrel{\text{def}}{=} -i \int_{\mathbb{R}} f u_j dx \int_{\mathbb{R}} g u_j dx.$$

The Lorentzians

$$\frac{|\text{Im } \lambda_j|}{|\lambda - \lambda_j|^2} = -2 \text{Im } \frac{1}{\lambda_j - \lambda},$$

peak at  $\lambda = \text{Re } \lambda_j$  and are more pronounced for  $\text{Im } \lambda_j$  small. This stronger response in the spectrum of correlations is one of the reasons for calling  $\lambda_j$  (or  $\lambda_j^2$ ) *scattering resonances*.

In more general situations, to have a finite expansion of type (1.6), modulo some exponentially decaying error  $\mathcal{O}(e^{-\gamma t})$ , we need to know that the number of poles of  $R(\lambda)$  is finite in a strip  $\text{Im } \lambda > -\gamma$ . Hence exponential decay of correlations is closely related to *resonance free strips*.

This elementary example is related through our approach to recent results of Dolgopyat [16], Liverani [29], and Tsujii [37] on the decay of correlations in classical dynamics.

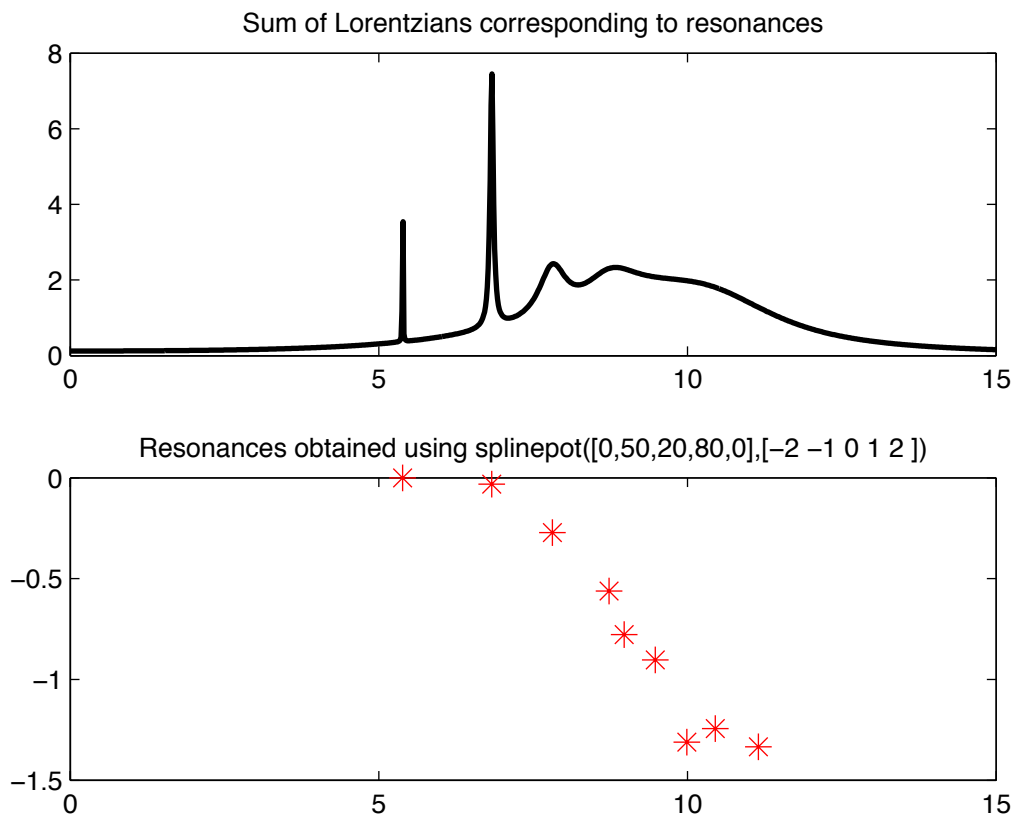


FIGURE 1. The effect of resonances on the Fourier transform of correlations as described in (1.7). The resonances are computed using the code `scatpot.m` [3].

Let  $X$  be a compact contact manifold of (odd) dimension  $n$ , and let  $\gamma_t$  be an Anosov flow on  $X$  preserving the contact structure — see §9 for details. The standard example is the geodesic flow on the cosphere bundle  $X = S^*M$ , where  $(M, g)$  is a smooth negatively curved Riemannian manifold. Let  $U(t) : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$  be defined by  $U(t)f = \gamma_t^* f = f \circ \gamma_t$  and let  $dx$  be the measure on  $X$  induced by the contact structure and normalized so that  $\text{vol}(X) = 1$ . The results of [16],[29],[37] show that, for any test functions  $f, g \in C^\infty(X)$ , the correlation function satisfies the following asymptotical behavior for large times:

$$(1.8) \quad C(f, g)(t) \stackrel{\text{def}}{=} \int_X [U(t)f](x) g(x) dx = \int_X f dx \int_X g dx + \mathcal{O}(e^{-\Gamma t}), \quad t \rightarrow \infty,$$

and the exponent  $\Gamma$  is independent of  $f, g$ . In other words, the Anosov flow is exponentially mixing with respect to the invariant measure  $dx$ .

From the microlocal point of view of Faure-Sjöstrand [21], this result is related to a resonance free strip for the generator of the flow  $\gamma_t$ . The resonances in this setting are called *Pollicott-Ruelle resonances*.

In this paper we consider general semiclassical operators modeled on  $P$  given in (1.1), for which the classical flow has a normally hyperbolic trapped set. Schrödinger operators for which (1.2) holds appear in molecular dynamics – see the recent review [26] for an introduction and references. In particular, [26, Chapter 5] discusses the resonances in some model cases and the relation between the size of the resonance free strip and Lyapounov exponents. As reviewed in §9, the setting can be extended such as to include the generator of the Anosov flow of (1.8), namely the operator  $P(h)$  on  $X$  such that  $U(t) = \gamma_t^* = \exp(-itP/h)$ .

**1.3. Assumptions and the result.** The general result, Theorem 2, is proved for operators modified using a *complex absorbing potential* (CAP). Results about such operators can then be used for different problems using *resolvent gluing techniques* of Datchev-Vasy [13] – see Theorems 3 and 4. The assumptions on the manifold  $X$ , operator  $P$ , and the complex absorbing potential may seem unduly general, they are justified by the broad range of applications.

Let  $X$  be a smooth compact manifold with a density  $dx$  and let

$$P = P(x, hD) \in \Psi^m(X),$$

be an unbounded self-adjoint semiclassical pseudodifferential operator on  $L^2(X, dx)$  (see §3.1 and [43, §14.2] for background and notations), with principal symbol  $p(x, \xi)$  independent of  $h$ . Let

$$W = W(x, hD) \in \Psi^k(X), \quad 0 \leq k \leq m, \quad W \geq 0,$$

be another operator, also self-adjoint and with  $h$ -independent principal symbol  $w(x, \xi)$ , which we call a (generalized) complex absorbing potential (CAP). We should stress that  $W$  plays a purely *auxiliary* role and can be chosen quite freely.

If the principal symbols  $p(x, \xi) \in S^m(T^*X)$  and  $w(x, \xi) \in S^k(T^*X)$ , we assume that, for some fixed  $C_0 > 0$  and for any phase space point  $(x, \xi) \in T^*X$ ,

$$(1.9) \quad \begin{aligned} |p(x, \xi) - iw(x, \xi)| &\geq \langle \xi \rangle^m / C_0 - C_0, \quad 1 + w(x, \xi) \geq \langle \xi \rangle^k / C_0, \\ \exp(tH_p)(x, \xi) &\text{ is defined for all } t \in \mathbb{R}. \end{aligned}$$

Here, for  $\xi \in T_x^*X$  we have denoted  $\langle \xi \rangle^2 = 1 + \|\xi\|_x^2$  for some smoothly varying metric on  $X$ ,  $x \mapsto \|\bullet\|_x^2$ , and by  $H_p$  the Hamilton vector field of  $p$ . The map  $\exp(tH_p) : T^*X \rightarrow T^*X$  is the corresponding flow at time  $t$ . This flow will often be denoted by  $\varphi_t$ , the Hamiltonian  $p(x, \xi)$  being clear from the context.

We will need an additional smoothness assumption on  $w$ :

$$(1.10) \quad |\partial^\alpha w(x, \xi)| \leq C_\alpha w(x, \xi)^{1-\gamma}, \quad 0 < \gamma < \frac{1}{2},$$

when  $w(x, \xi) \leq 1$ . This can be easily arranged and is invariant under changes of variables.

We call the operator

$$(1.11) \quad \tilde{P} = P - iW \in \Psi^m(X),$$

the CAP-modified  $P$ . The condition (1.9) means that the CAP-modified  $P$  is classically elliptic and that for any fixed  $z \in \mathbb{C}$

$$\{(x, \xi) : \tilde{p}(x, \xi) - z = p(x, \xi) - iw(x, \xi) - z = 0\} \Subset T^*X.$$

We define the *trapped set* at energy  $E$  as

$$(1.12) \quad K_E \stackrel{\text{def}}{=} \{\rho = (x, \xi) : \rho \in p^{-1}(E), \varphi_{\mathbb{R}}(\rho) \subset w^{-1}(0)\}.$$

$K_E$  is compact and consists of points in  $p^{-1}(E)$  which never reach the *damping region*  $\{\rho \in T^*X : w(\rho) > 0\}$  in backward or forward propagation by the flow  $\varphi_t$ .

We illustrate this setup with two simple examples:

**Example 1.** Suppose that  $P_0 = -h^2\Delta + V$ ,  $V \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R})$ ,  $\text{supp } V \Subset B(0, R_0)$ . Define the torus  $X = \mathbb{R}^n / (3R_0\mathbb{Z})^n$ , and  $W \in \mathcal{C}^\infty(X; [0, \infty))$ , satisfying

$$W(x) = 0, \quad x \in B(0, R_0), \quad W(x) = 1, \quad x \in X \setminus B(0, 2R_0), \quad \partial^\alpha W = \mathcal{O}(W^{2/3}),$$

(here we identified the balls in  $\mathbb{R}^n$  with subsets of the torus). The last condition can be arranged by taking  $W(x) = \chi(|x|^2 - R_0)\psi(x)$  where  $\chi(x) = \exp(-x^{-1})x_+^0$ , and  $\psi \in \mathcal{C}^\infty(X, (0, \infty))$  is suitably chosen. The power of  $W$  on the right hand side can be any number greater than  $\frac{1}{2}$ .

Because of the support properties of  $V$ ,  $P \stackrel{\text{def}}{=} -h^2\Delta + V \in \Psi^2(X)$  and  $P - iW$  satisfy all the properties above. The trapped set  $K_E$  can be identified with a subset of  $T_{B(0, R_0)}^*\mathbb{R}^n$  and is then equal to the trapped set of scattering theory:

$$K_E = \{(x, \xi) \in T^*\mathbb{R}^n : \xi^2 + V(x) = E, \quad x(t) \not\rightarrow \infty, \quad t \rightarrow \pm\infty\}.$$

**Example 2.** Suppose that  $X$  is a compact manifold with a volume form  $dx$  and a vector field  $\Xi$  generating a volume preserving flow ( $\mathcal{L}_\Xi dx = 0$ ). Then  $P = -ih\Xi$  is a self-adjoint operator on  $L^2(X, dx)$ , and the corresponding propagator  $\exp(-itP/h)$  is the push-forward of the flow  $\gamma_t = \exp(t\Xi)$  generated by  $\Xi$  on functions  $f \in L^2(X, dx)$ :  $\exp(-itP/h)f = f \circ \gamma_{-t}$ .

To define the CAP in this setting we choose a Riemannian metric  $g$  on  $X$ , and a function

$$(1.13) \quad \begin{aligned} f &\in \mathcal{C}^\infty(\mathbb{R}, [0, \infty)), \quad |f^{(k)}(s)| \leq C f(s)^{1-\gamma}, \quad 0 < \gamma < \frac{1}{2}, \\ f^{-1}(0) &= [-\infty, M] \text{ for some } M > 0, \quad f(s) = \sqrt{s}, \quad s > 2M. \end{aligned}$$

If  $\Delta_g$  is the corresponding Laplacian on  $X$ , we set  $W(x, hD) = f(-h^2\Delta_g)$ .

Then the operator  $P - iW \in \Psi^1(X)$  satisfies the assumptions above. The principal symbols  $p(x, \xi) = \xi(\Xi_x)$ ,  $w(x, \xi) = f(\|\xi\|_x^2)$ , where the norm  $\|\bullet\|_x$  is associated with the metric  $g$ .

The trapped set is given by the points which never enter the damping region:

$$K_E = \{(x, \xi) \in T^*X : \xi(\Xi_x) = E, \ \|(\gamma_{-t})_*\xi\|_g \leq M, \ \forall t \in \mathbb{R}\}.$$

At this stage the trapped set seems to depend on the choice of  $M$ . Below we will be concerned with  $\exp(t\Xi)$  being an Anosov flow, in which case this explicit dependence will disappear.

Returning to general considerations we also define

$$(1.14) \quad K^\delta \stackrel{\text{def}}{=} \bigcup_{|E| \leq \delta} K_E,$$

which is a compact subset to  $T^*X$  and assume that

$$(1.15) \quad dp|_{K^\delta} \neq 0, \quad K^\delta \cap \text{WF}_h(W) = \emptyset.$$

The first assumption implies that for  $|E| \leq \delta$ , the energy shell  $p^{-1}(E)$  is a smooth hypersurface close to  $w^{-1}(0)$ . The second assumption is motivated by applications: in scattering problems the CAP is introduced only away from the trapped set. This second assumption also means that the trapped set is unchanged upon small modifications of  $\text{WF}_h(W)$ .

We now make the following *normal hyperbolicity* assumption on  $K^\delta$ :

$$(1.16) \quad K^\delta \text{ is a smooth symplectic submanifold of } T^*X,$$

and there exists a continuous distribution of linear subspaces

$$K^\delta \ni \rho \longmapsto E_\rho^\pm \subset T_\rho(T^*X),$$

invariant under the flow,

$$\forall t \in \mathbb{R}, \quad (\varphi_t)_* E_\rho^\pm = E_{\exp t H_p(\rho)}^\pm,$$

and satisfying, for some  $\lambda > 0$ ,  $C > 0$  and any point  $\rho \in K^\delta$ ,

$$(1.17) \quad \begin{aligned} T_\rho K^\delta \cap E_\rho^\pm &= E_\rho^+ \cap E_\rho^- = \{0\}, \quad \dim E_\rho^\pm = d_t, \quad T_\rho(T^*X) = T_\rho K^\delta \oplus E_\rho^+ \oplus E_\rho^-, \\ \forall v \in E_\rho^\pm, \quad \forall t > 0, \quad \|d\varphi_{\mp t}(\rho)v\|_{\varphi_{\mp t}(\rho)} &\leq C e^{-\lambda t} \|v\|_\rho. \end{aligned}$$

Here  $\rho \mapsto \|\bullet\|_\rho$  is any smoothly varying norm on  $T_\rho(T^*X)$ ,  $\rho \in K^\delta$ . The choice of norm may affect  $C$  but not  $\lambda$ .

We can now state our main result.

**Theorem 2.** *Suppose that  $X$  is a smooth compact manifold and that  $P$  and  $W$  satisfy the assumptions above. If the trapped set  $K^\delta$  given by (1.12), (1.14) is normally hyperbolic, in the sense that (1.16) and (1.17) hold, then for any  $\epsilon > 0$  there exists  $h_0$ ,  $c_0$ ,  $C_1$ , such that for  $0 < h < h_0$ ,*

$$(1.18) \quad \begin{aligned} \|(P - iW - z)^{-1}\|_{L^2 \rightarrow L^2} &\leq C_1 h^{-1+c_0 \text{Im } z/h} \log(1/h), \\ \text{for } z &\in [-\delta + \epsilon, \delta - \epsilon] - ih[0, \lambda_0/2 - \epsilon], \end{aligned}$$



where

$$(1.19) \quad \lambda_0 \stackrel{\text{def}}{=} \liminf_{t \rightarrow \infty} \frac{1}{t} \inf_{\rho \in K^\delta} \log \det \left( d\varphi_t|_{E_\rho^+} \right).$$

Here  $\det$  is taken using any fixed volume form on  $E_\rho^+$ , the value of  $\lambda_0 > 0$  being independent of the choice of volume forms.

This theorem will be proved in §6 after preparation in §§4,5. The bound  $\log(1/h)/h$  on the real axis is optimal as shown in [4] and [15]. Using the methods of [13] the estimate (1.18) almost immediately applies to the setting of scattering theory. As an example we present an application to scattering on asymptotically hyperbolic manifolds, which will be proved in §8:

**Theorem 3.** *Suppose  $(Y, g)$  is a conformally compact  $n$ -manifold with even power metric:  $Y$  is compact,  $\partial Y = \{x = 0\}$ ,  $dx|_{\partial Y} \neq 0$ ,  $g = (dx^2 + h)/x^2$  where  $h$  is a smooth 2-tensor on  $Y$  with only even powers of  $x$  appearing in its Taylor expansion at  $x = 0$ . If the trapped set for the geodesic flow on  $Y$  is normally hyperbolic, then the following resolvent estimate holds:*

$$\|x^{k_0} (-\Delta_g - (n-1)^2/4 - \lambda^2)^{-1} x^{k_0}\|_{L^2 \rightarrow L^2} \leq C_0 \frac{\log \lambda}{\lambda}, \quad \lambda > 1.$$

The next application is a rephrasing of a recent theorem of Tsujii [37]; it will be proved in §9. We take the point of view of Faure-Sjöstrand [21], see also [12].

**Theorem 4.** *Suppose  $X$  is a compact manifold and  $\gamma_t : X \rightarrow X$  a contact Anosov flow on  $X$ . Let  $\Xi$  be the vector field generating  $\varphi_t$ , and  $P = -ih\Xi$  the corresponding semiclassical operator, self-adjoint on  $L^2(X, dx)$  for  $dx$  the volume form derived from the contact structure.*

*Define the minimal asymptotic unstable expansion rate*

$$(1.20) \quad \lambda_0 \stackrel{\text{def}}{=} \liminf_{t \rightarrow \infty} \frac{1}{t} \inf_{x \in X} \log \det \left( d\gamma_t|_{E_u(x)} \right),$$

*with  $E_u(x) \subset T_x X$  the unstable subspace of the flow at  $x$ .*

*For any  $t > 0$  there exists a Hilbert space,  $H_{tG}$  (see (9.10)),*

$$\mathcal{C}^\infty(X) \subset H_{tG}(X) \subset \mathcal{D}'(X)$$

*such that  $(P - z)^{-1} : H_{tG} \rightarrow H_{tG}$  is meromorphic for  $\text{Im } z > -th$ .*

*Then for any small  $\epsilon, \delta > 0$ , there exist  $h_0, c_0 > 0$  and  $C_1 > 0$  such that, taking any  $t > \lambda_0/2$  and any  $0 < h < h_0$ ,*

$$(1.21) \quad \|(P - z)^{-1}\|_{H_{tG} \rightarrow H_{tG}} \leq C_1 h^{-1+c_0 \text{Im } z/h} \log(1/h), \quad z \in [\delta, \delta^{-1}] - ih[0, \lambda_0/2 - \epsilon].$$

We remark that under a pinching condition on the Lyapunov exponents, the recent results announced by Faure-Tsujii [23] provide a much more precise description of the spectrum of  $P = -ih\Xi$  on  $H_{tG}$ : the Ruelle-Pollicott resonances are localized in horizontal strips below

the real axis, and the number of resonances in each strip satisfies a Weyl's law asymptotics. That is analogous to the result proved by Dyatlov [19], which was motivated by quasinormal modes for black holes.

Theorems 3 and 4 have applications to the decay of correlations, respectively for the wave equation and for contact Anosov flows. As an example we state a refinement of the decay of correlation result (1.8) of Dolgopyat [16], Liverani [29] and Tsujii [37]:

**Theorem 5.** *Suppose that  $\gamma_t : X \rightarrow X$  is a contact Anosov flow on a compact manifold  $X$  (see §9.1 for the definitions) and that  $\lambda_0$  is given by (1.20).*

*Then there exist a sequence of complex numbers,  $\mu_j$ ,*

$$0 > \operatorname{Im} \mu_j \geq \operatorname{Im} \mu_{j+1},$$

*and of distributions  $u_{j,k}, v_{j,k} \in \mathcal{D}'(X)$ ,  $0 \leq k \leq K_j$ , such that, for any  $\epsilon > 0$ , there exists  $J(\epsilon)$  such that for any  $f, g \in \mathcal{C}^\infty(X)$ ,*

$$(1.22) \quad \int_X f(x) \gamma_t^* g(x) dx = \int_X f dx \int_X g dx + \sum_{j=1}^{J(\epsilon)} \sum_{k=1}^{K_j} t^k e^{-it\mu_j} u_{j,k}(f) v_{j,k}(g) + \mathcal{O}_{f,g}(e^{-t(\lambda_0 - \epsilon)/2}),$$

*for  $t > 0$ . Here  $dx$  is the measure on  $X$  induced by the contact form and normalized so that  $\operatorname{vol}(X) = 1$ , and  $u(f)$ ,  $u \in \mathcal{D}'(X)$ ,  $f \in \mathcal{C}^\infty(X)$  denotes the distributional pairing.*

For information about microlocal structure of the distributions  $u_{j,k}$  and  $v_{j,k}$  the reader should consult [21]. Here we only mention that (with the standard wave front set of [27])

$$\operatorname{WF}(u_{j,k}) \subset E_u^*, \quad \operatorname{WF}(\bar{v}_{j,k}) \subset E_s^*,$$

where  $E_\bullet^* = \bigcup_{x \in X} E_\bullet^*(x)$ , and  $E_\bullet^*(x) \subset T_x^*X$  is the annihilator of  $\mathbb{R}\Xi_x + E_\bullet(x) \subset T_x X$ ,  $\bullet = u, s$ . The spaces  $E_\bullet(x)$  appear in the Anosov decomposition of the tangent space (9.2)

## 2. OUTLINE OF THE PROOF OF THEOREM 2

The proof proceeds via the analysis of the propagator for the operator

$$\tilde{P}_G \stackrel{\text{def}}{=} e^{-G^w(x, hD)} (P - iW) e^{G^w(x, hD)},$$

where the function  $G(x, \xi; h)$  belongs to a certain exotic class of symbols. Our  $G$  which is closely related to the escape function constructed in [31], it depends on an additional small parameter,  $\tilde{h}$ , which will be chosen independently of  $h$ .

For a large  $t_0$ , any fixed  $\Gamma > 0$  and  $\epsilon > 0$ , we can construct  $G$  so that, for some constant  $C_0$ , the following holds uniformly in  $0 < h < h_0$ ,  $0 < \tilde{h} < \tilde{h}_0$ :

$$(2.1) \quad \begin{aligned} G(\rho) &= \mathcal{O}(\log(1/h)), \quad G(\rho) - G(\varphi_{-t_0}(\rho)) \geq -C_0, \quad \rho \in T^*X, \\ G(\rho) - G(\varphi_{-t_0}(\rho)) &\geq 2\Gamma, \quad \rho \in p^{-1}([-\delta, \delta]), \quad d(\rho, K^\delta) > (h/\tilde{h})^{\frac{1}{2}}, \quad w(\rho) < \epsilon, \end{aligned}$$

where  $d(\bullet, \bullet)$  is any given distance function in  $T^*X$ .

The proof of Theorem 2 is based on the following estimate. For some  $\epsilon_0 > 0$ , take an operator  $A \in \Psi^0(X)$  such that  $\text{WF}_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0))$ . We will prove the norm estimate

$$(2.2) \quad \|\exp(-it\tilde{P}_G/h)A\|_{L^2(X) \rightarrow L^2(X)} \leq C e^{-t(\lambda_0 - \epsilon)/2}, \quad M_\epsilon \log \frac{1}{\tilde{h}} \leq t \leq M \log \frac{1}{\tilde{h}},$$

As a result, for  $\text{Im } z > -(\lambda_0 - 2\epsilon)/2$ ,

$$(2.3) \quad (\tilde{P}_G - z) \frac{i}{h} \int_0^T e^{-it(\tilde{P}_G - z)/h} A dt = (I - e^{-iT(\tilde{P}_G - z)/h})A = A - \mathcal{O}(e^{-T\epsilon})_{L^2 \rightarrow L^2}.$$

Hence, by taking  $T$  large enough and using the ellipticity of  $\tilde{P}_G - z$  away from  $p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0))$ , we obtain

$$(\tilde{P}_G - z)^{-1} = \mathcal{O}(h^{-1})_{L^2 \rightarrow L^2}, \quad \text{Im } z > -(\lambda_0 - 2\epsilon)/2.$$

Since  $e^{\pm G^w} = \mathcal{O}(h^{-M})_{L^2 \rightarrow L^2}$  from the growth condition on  $G$ , a polynomial bound for  $(P - iW - z)^{-1}$  follows. The more precise bound (1.18) follows from a semiclassical maximum principle.

To prove the estimate (2.2) we proceed in a number of steps:

**Step 1.** The most delicate part of the argument concerns the evolution near the trapped set. For some fixed  $R > 1$ , we introduce a cut-off function  $\chi \in \tilde{S}_{\frac{1}{2}}$  supported in the set

$$\{\rho \in p^{-1}((-\delta, \delta)) : d(\rho, K^\delta) \leq 2R(h/\tilde{h})^{\frac{1}{2}}\}.$$

This cut-off is quantized into an operator  $\chi^w \stackrel{\text{def}}{=} \chi^w(x, hD)$ .

We then claim that

for any  $\epsilon > 0$  and  $M > 0$ , there exists  $C > 0$  such that

$$(2.4) \quad \|\chi^w e^{-itP/h} \chi^w\|_{L^2 \rightarrow L^2} \leq C \tilde{h}^{-dt/2} e^{-t(\lambda_0 - \epsilon)/2} + \mathcal{O}(\tilde{h}^\infty),$$

uniformly for  $0 \leq t \leq M \log \frac{1}{\tilde{h}}$ .

The proof of this bound is provided in §5.

**Step 2.** For the weighted operator we obtain an improved estimate, now with a fixed large time  $t_0$  related to the construction of  $G$ , and for  $\chi$  which in addition satisfies

$$\chi(\rho) = 1 \quad \text{for } d(\rho, K^\delta) \leq R(h/\tilde{h})^{\frac{1}{2}}, \quad |p(\rho)| \leq \delta/2.$$

Using Egorov's theorem and (from (2.1)) the positivity of  $G - G \circ \varphi_{-t_0}$  on the set  $\text{supp}(1 - \chi) \cap \text{WF}_h(A)$ , we get following the weighted estimate:

$$(2.5) \quad \|(1 - \chi^w) e^{-it_0 \tilde{P}_G/h} A\| \leq e^{-\Gamma},$$

When constructing the function  $G$  it is essential to choose  $\Gamma$  such that

$$\Gamma > \frac{t_0 \lambda_0}{2}.$$

We also show that

$$(2.6) \quad \|e^{-it_0 \tilde{P}_G/h} A\| \leq e^{2C_0},$$

for a constant  $C_0$  independent of  $h, \tilde{h}$ . Formally, these results follow from Egorov's theorem but care is needed as  $G$  is a symbol in an exotic class. To obtain (2.5) and (2.6) we proceed as in the proof of [31, Proposition 3.11]. This is done in §6.

**Step 3.** The last step combines the two previous estimates, by decomposing

$$\begin{aligned} e^{-int_0 \tilde{P}_G/h} &= (U_{G,+} + U_{G,-})^n, \\ U_{G,+} &\stackrel{\text{def}}{=} e^{-it_0 \tilde{P}_G/h} \chi^w, \quad U_{G,-} \stackrel{\text{def}}{=} e^{-it_0 \tilde{P}_G/h} (1 - \chi^w). \end{aligned}$$

In order to apply (2.4) we use the fact that

$$\chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} \chi^w = \chi_{G,1}^w e^{-itP/h} \chi_{G,2}^w + \mathcal{O}(\tilde{h}^\infty) + \mathcal{O}(h^{\frac{1}{2}}),$$

where the symbols  $\chi_{G,i}$  have the properties required in (2.4). A clever expansion of  $(e^{int_0 \tilde{P}_G/h})^n$  into terms involving  $U_{G,\pm}$  and an application of Steps 1 and 2 lead to the estimate (2.2) for  $t = nt_0$ . The argument is presented in §7.

### 3. PRELIMINARIES

In this section we will briefly recall basic concepts of semiclassical quantization on manifolds with detailed references to previous papers.

**3.1. Semiclassical quantization.** The semiclassical pseudodifferential operators on a compact manifold  $X$  are quantizations of functions belonging to the symbol classes  $S^m$  modeled on symbol classes for  $\mathbb{R}^n$ :

$S^m(T^*\mathbb{R}^n) = \{a \in \mathcal{C}^\infty(T^*\mathbb{R}^n \times (0, 1]_h) : \forall \alpha, \beta \in \mathbb{N}^n, |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|}\},$   
see [43, §14.2.3]. The Weyl quantization, which we informally write as

$$S^m(T^*X) \ni a(x, \xi) \longmapsto a^w(x, hD) \in \Psi^m(X),$$

maps symbols to pseudodifferential operators. It is modeled on the quantization on  $\mathbb{R}^n$ :

$$(3.1) \quad \begin{aligned} [a^w u](x) &= a^w(x, hD)u(x) = [\text{Op}_h^w(a)u](x) \\ &\stackrel{\text{def}}{=} \frac{1}{(2\pi h)^d} \int \int a\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y, \xi \rangle/h} u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

The symbol map

$$\sigma : \Psi^m(X) \rightarrow S^m(T^*X)/hS^{m-1}(T^*X),$$

is well defined as an equivalence class and its kernel is  $h\Psi^{m-1}(X)$  – see [43, Theorem 14.3]. If  $\sigma(A)$  has a representative independent of  $h$  we call that invariantly defined element of  $S^m(T^*X)$  the *principal symbol* of  $A$ .

Following [11] we define the class of *compactly microlocalized operators*

$$\Psi^{\text{comp}}(X) \stackrel{\text{def}}{=} \{a^w(x, hD) : a \in (S^0 \cap \mathcal{C}_c^\infty)(T^*X)\} + h^\infty \Psi^{-\infty}(X).$$

These operators have well defined semiclassical wave front sets:

$$\Psi^{\text{comp}}(X) \ni A \longmapsto \text{WF}_h(A) \Subset T^*X,$$

see [11, §3.1] and [43, §8.4].

Let  $u = u(h)$ ,  $\|u(h)\|_{L^2} = \mathcal{O}(h^{-N})$  (for some fixed  $N$ ) be a wavefunction microlocalized in a compact set in  $T^*X$ , in the sense that for some  $A \in \Psi^{\text{comp}}$ , one has  $u = Au + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty)$ . The semiclassical wavefront set of  $u$  is then defined as:

$$(3.2) \quad \text{WF}_h(u) = \mathfrak{C}\{\rho \in T^*X : \exists a \in S^0(T^*X), \ a(x, \xi) = 1, \ \|a^w u\|_{L^2} = \mathcal{O}(h^\infty)\}.$$

When  $A \in \Psi^{\text{comp}}(X)$  we also define

$$\text{WF}_h(I - A) := \bigcup_{B \in \Psi^{\text{comp}}(X)} \text{WF}_h(B(I - A)),$$

and note that  $\text{WF}_h(B) \cap \text{WF}_h(A)$  is defined for any  $B \in \Psi^m(X)$  as

$$\text{WF}_h(B) \cap \text{WF}_h(A) := \text{WF}_h(CB) \cap \text{WF}_h(A), \quad C \in \Psi^{\text{comp}}, \quad \text{WF}_h(I - C) \cap \text{WF}_h(A) = \emptyset.$$

Semiclassical Sobolev spaces,  $H_h^s(X)$  are defined using the norms

$$(3.3) \quad \|u\|_{H_h^s(X)} = \|(I - h^2 \Delta_g)^{s/2} u\|_{L^2(X)},$$

for some choice of Riemannian metric  $g$  on  $X$  (notice that  $H_h^s(X)$  represents the same vector space as the usual Sobolev space  $H^s(X)$ ).

**3.2.  $S_{\frac{1}{2}}$  calculus with two parameter.** Another standard space of symbols  $S_\delta(\mathbb{R}^{2n})$ ,  $0 < \delta \leq 1/2$ , is defined by demanding that  $\partial^\alpha a = \mathcal{O}(h^{-|\alpha|\delta})$ . The quantization procedure  $a \mapsto \text{Op}_h^w a$  gives well defined operators and  $\text{Op}_h^w a \circ \text{Op}_h^w b = \text{Op}_h^w c$  with  $c \in S_\delta$ .

For  $0 < \delta < 1/2$  we still have an pseudodifferential calculus, with asymptotic expansions in powers of  $h$ . However, for  $\delta = 1/2$  we are at the border of the uncertainty principle, and there is no asymptotic calculus – see [43, §4.4.1]. To obtain an asymptotic calculus the standard  $S_{\frac{1}{2}}$  spaces is replaced by a symbol space where a second asymptotic parameter is introduced:

$$\tilde{S}_{\frac{1}{2}}(\mathbb{R}^{2n}) \stackrel{\text{def}}{=} \{a = a(\rho, h, \tilde{h}) \in \mathcal{C}^\infty(\mathbb{R}_\rho^{2n} \times (0, 1]_h \times (0, 1]_{\tilde{h}}) : |\partial_\rho^\alpha a| \leq C_\alpha (h/\tilde{h})^{-|\alpha|/2}\}.$$

Then the quantization  $a \mapsto a^w(x, hD) \in \tilde{\Psi}_{\frac{1}{2}}(\mathbb{R}^n)$  is unitarily equivalent to

$$(3.4) \quad \tilde{a} \mapsto \tilde{a}^w(\tilde{x}, \tilde{h}D) = \text{Op}_{\tilde{h}}^w(\tilde{a}), \quad \tilde{a}(\rho) = a((h/\tilde{h})^{\frac{1}{2}}\rho), \quad \tilde{a} \in S(\mathbb{R}^{2n}),$$

— see [43, §§4.1.1, 4.7.2]. Hence, we now have expansions in powers of  $\tilde{h}$ , as in the standard calculus, with better properties (powers of  $(h\tilde{h})^{\frac{1}{2}}$ ) when operators in  $\tilde{\Psi}_{\frac{1}{2}}$  and  $\Psi$  are composed — see [35, Lemma 3.6].

For the case of manifolds we refer to [11, §5.1] which generalizes and clarifies the presentations in [35, §3.3] and [42, §3.2]. The basic space of symbols, and the only one needed here, is

$$\begin{aligned} \tilde{S}_{\frac{1}{2}}^{\text{comp}}(T^*X) = & \{a \in \mathcal{C}_c^\infty(T^*X) : V_1 \cdots V_k a = \mathcal{O}((h/\tilde{h})^{-\frac{k}{2}}), \quad \forall k, \\ & V_j \in \mathcal{C}^\infty(T^*X, T(T^*X))\} + h^\infty S^{-\infty}(T^*X). \end{aligned}$$

The quantization procedure

$$\tilde{S}_{\frac{1}{2}}^{\text{comp}}(T^*X) \ni a \rightarrow \text{Op}_h^w(a) \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X)$$

defines the class of operators  $\tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X)$  modulo  $h^\infty \Psi^{-\infty}(X)$ , and the symbol map:

$$(3.5) \quad \tilde{\sigma} : \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X) \longrightarrow \tilde{S}_{\frac{1}{2}}^{\text{comp}}(T^*X)/h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}\tilde{S}_{\frac{1}{2}}^{\text{comp}}(T^*X).$$

The properties of the resulting calculus are listed in [11, Lemma 5.1] and we will refer to those results later on.

When  $\tilde{h} = 1$  we use the notation  $S_{\frac{1}{2}}^{\text{comp}}(T^*X)$  for symbols and denote by  $\Psi_{\frac{1}{2}}^{\text{comp}}(X)$  the corresponding class of pseudodifferential operators. The symbol map

$$\sigma : \Psi_{\frac{1}{2}}^{\text{comp}}(X) \longrightarrow S_{\frac{1}{2}}^{\text{comp}}(T^*X)/h^{\frac{1}{2}}S_{\frac{1}{2}}^{\text{comp}}(T^*X),$$

is still well defined but the operators in this class do not enjoy a proper symbol calculus in the sense that  $\sigma(AB)$  cannot be related to  $\sigma(A)\sigma(B)$ . However, when  $A \in \Psi_{\frac{1}{2}}^{\text{comp}}(X)$  and  $B \in \Psi(X)$  then  $\sigma(AB) = \sigma(A)\sigma(B) + \mathcal{O}(h^{\frac{1}{2}})_{S_{\frac{1}{2}}^{\frac{1}{2}}(T^*X)}$  — see [35, Lemma 3.6] or [11, Lemma 5.1].

**3.3. Fourier integral operators.** In this paper we will consider Fourier integral operators associated to canonical transformations. It will also be sufficient to consider operators which are compactly microlocalized as we will always work near  $p^{-1}([-2\delta, 2\delta]) \cap w^{-1}(0)$  which by assumption (1.9) is a compact subset of  $T^*X$ .

Suppose that  $Y_1, Y_2$  are two compact smooth manifolds ( $Y_j = X$  or  $Y_j = \mathbb{T}^n$  in what follows) and that,  $U_j \subset T^*Y_j$  are open subsets. Let

$$\kappa : U_1 \rightarrow U_2, \quad \Gamma_\kappa \stackrel{\text{def}}{=} \{(x, \xi, y, -\eta) : (x, \xi) = \kappa(y, \eta), (y, \eta) \in U_1\} \subset T^*Y_2 \times T^*Y_1,$$

be a symplectic transformation, for instance  $\kappa = \varphi_t$ ,  $U_1 = U_2 = T^*X$ . Here  $\Gamma_\kappa$  is the graph of  $\kappa$  and  $'$  denotes the twisting  $\eta \mapsto -\eta$ . This follows the standard convention [27, Chapter 25].

Following [11, §5.2] we introduce the class of *compactly microlocalized  $h$ -Fourier integral operator quantizing  $\kappa$* ,  $I_h^{\text{comp}}(Y_2 \times Y_1, \Gamma'_\kappa)$ . If  $T \in I_h^{\text{comp}}(Y_2 \times Y_1, \Gamma'_\kappa)$  then it has the following properties:  $T = \mathcal{O}(1)_{L^2(Y_1) \rightarrow L^2(Y_2)}$ ; there exist  $A_j \in \Psi^{\text{comp}}(Y_j)$ ,  $\text{WF}_h(A_j) \subseteq U_j$  such that

$$A_2 T = T + \mathcal{O}(h^\infty)_{\mathcal{D}'(Y_1) \rightarrow \mathcal{C}^\infty(Y_2)}, \quad T A_1 = T + \mathcal{O}(h^\infty)_{\mathcal{D}'(Y_1) \rightarrow \mathcal{C}^\infty(Y_2)};$$

for any  $B_j \in \Psi^m(Y_j)$ ,

$$(3.6) \quad \begin{aligned} T B_1 &= C_1 T + h T_1, \quad \sigma(C_1) = \sigma(B_1) \circ \kappa^{-1}, \\ B_2 T &= T C_2 + h T_2, \quad \sigma(C_2) = \sigma(B_2) \circ \kappa, \quad T_j \in I_h^{\text{comp}}(Y_2 \times Y_1, \Gamma'_\kappa). \end{aligned}$$

The last statement is a form of Egorov theorem.

When  $B_j \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X)$  then an analogue of (3.6) still holds in a modified form

$$(3.7) \quad \begin{aligned} T B_1 &= C_1 T + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} D_1 T_1, \quad \sigma(C_1) = \sigma(B_1) \circ \kappa^{-1}, \\ B_2 T &= T C_2 + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} T_2 D_2, \quad \sigma(C_2) = \sigma(B_2) \circ \kappa, \\ T_j &\in I_h^{\text{comp}}(Y_2 \times Y_1, \Gamma'_\kappa), \quad C_j, D_j \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X), \end{aligned}$$

see Proposition 6.3 (applied with  $g \equiv 0$ ).

An example is given by the operators

$$(3.8) \quad A e^{-itP/h}, \quad e^{-itP/h} A \in I^{\text{comp}}(X \times X, \Gamma'_{\varphi_t}), \quad \text{if } A \in \Psi^{\text{comp}}(X).$$

In §5 we will also need a local representation of elements of  $I^{\text{comp}}$  as oscillatory integrals – see [1], [20, §3.2] and references given there. If  $T \in I^{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma'_\kappa)$  is microlocalized to a sufficiently small neighbourhood  $\kappa(U) \times U \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$  ([43, 8.4.5]) then

$$(3.9) \quad T u(x) = (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{\frac{i}{h} \psi(x, y, \theta)} a(x, y, \theta) u(y) dy d\theta + \mathcal{O}(h^\infty)_S \|u\|_{H^{-M}},$$

for any  $M$ . Here  $a \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}^k)$ ,  $\psi(x, y, \theta) \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times \mathbb{R}^k)$ , and near  $\kappa(U) \times U$ , the graph of  $\kappa$  is given by

$$(3.10) \quad \begin{aligned} \Gamma_\kappa &= \{((x, d_x \psi(x, y, \theta)), (y, -d_y \psi(x, y, \theta))) : (x, y, \theta) \in C_\psi\}, \\ C_\psi &\stackrel{\text{def}}{=} \{(x, y, \theta) : d_\theta \psi(x, y, \theta) = 0, \}, \\ d_{x, y, \theta}(\partial_{\theta_j} \psi), \quad j &= 1, \dots, k, \quad \text{are linearly independent,} \end{aligned}$$

For given symplectic coordinates  $(x, \xi)$  and  $(y, \eta)$  in neighbourhoods of  $\kappa(U)$  and  $U$  respectively, such a representation exists with an extra variable of dimension  $k$ , where  $0 \leq k \leq n$ , and  $n + k$  is equal to the rank of the projection

$$\Gamma_\kappa \ni ((x, \xi), (y, \eta)) \mapsto (x, \eta),$$

assumed to be constant in the neighbourhood of  $\kappa(U) \times U$  – see for instance [43, Theorem 2.14]. Since  $\Gamma_\kappa$  in (3.10) is the graph of a symplectomorphism it follows that for some

$$y' = (y_{j_1}, \dots, y_{j_{n-k}}) \in \mathbb{R}^{n-k},$$

$$(3.11) \quad D_\psi(x, y, \theta) \stackrel{\text{def}}{=} \det \left( \frac{\partial^2 \psi}{\partial x_i \partial y'_{j'}}, \frac{\partial^2 \psi}{\partial x_i \partial \theta_j} \right) \neq 0.$$

For the use in §5 we record the following lemma, proved using standard arguments (see for instance [1]):

**Lemma 3.1.** *Suppose that  $T$  is given by (3.9) and that  $B \in \widetilde{\Psi}_{\frac{1}{2}}(\mathbb{R}^n)$ . Then for any  $u \in L^2$  with  $\|u\|_{L^2} = 1$ ,*

$$\begin{aligned} BTu(x) &= (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x, y, \theta) b(x, d_x \psi(x, y, \theta)) u(y) dy d\theta + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2}, \\ TBu(x) &= (2\pi h)^{-\frac{n+k}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x, y, \theta) b(y, -d_y \psi(x, y, \theta)) u(y) dy d\theta + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2}, \end{aligned}$$

where  $b = \sigma(B)$ .

**3.4. Fourier integral operators with operator valued symbols.** In §5 we will also use a class of Fourier integral operators with *operator valued* symbols. We present what we need in an abstract form in this section. Only local aspects of the theory will be relevant to us and we opt for a direct presentation.

Suppose that  $\mathcal{H}$  is a Hilbert space and  $Q$  is an (unbounded) self-adjoint operator with domain  $\mathcal{D} \subset \mathcal{H}$ . We assume that  $Q : \mathcal{D} \rightarrow \mathcal{H}$  is invertible and we put  $\mathcal{D}^\ell \stackrel{\text{def}}{=} Q^{-\ell} \mathcal{H}$ , for  $\ell \geq 0$ . For  $\ell < 0$ , we define  $\mathcal{D}^\ell$  as the completion of  $\mathcal{H}$  with respect to the norm  $\|Q^\ell u\|_{\mathcal{H}}$ .

We define the following class of operator valued symbols:

$$(3.12) \quad \mathcal{S}_\delta(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathcal{H}, \mathcal{D}),$$

to consist of operator valued functions

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \ni (x, y, \theta) \mapsto N(x, y, \theta) : \mathcal{D}^\infty \longrightarrow \mathcal{H}$$

which satisfy the following estimates:

$$(3.13) \quad \partial_{x,y,\theta}^\alpha N(x, y, \theta) = \mathcal{O}_{\alpha,\ell}(1) : \mathcal{D}^{\ell+|\delta||\alpha|} \longrightarrow \mathcal{D}^\ell,$$

for any multiindex  $\alpha$  and  $\ell \in \mathbb{Z}$ , uniformly in  $(x, y, \theta)$ . We note that this class is closed under pointwise composition of the operators: if  $N_j \in \mathcal{S}_\delta$  then  $N_j$  defines a family of operators  $\mathcal{D}^\ell \rightarrow \mathcal{D}^\ell$ , hence so does their product  $N_1 N_2$ ; the estimate (3.13) follows for the composition, since for  $|\beta| + |\gamma| = |\alpha|$ ,

$$\partial^\beta N_1 \partial^\gamma N_2 = \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta|\alpha|} \rightarrow \mathcal{D}^{\ell+\delta(|\alpha|-|\beta|)}} \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta(|\alpha|-|\beta|)} \rightarrow \mathcal{D}^\ell} = \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta|\alpha|} \rightarrow \mathcal{D}^\ell}.$$

Proposition 3.5 at the end of this section describes a class which will be used in §5.

Suppose that  $\psi$  satisfies (3.10) and (3.11). We can assume that  $\psi$  is defined on  $\mathbb{R}^{2n} \times \mathbb{R}^k$ . For  $N \in \mathcal{S}_\delta$  and  $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2n} \times \mathbb{R}^k)$  we define the operator

$$T : L^2(\mathbb{R}^n) \otimes \mathcal{H} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{H},$$



by

$$(3.14) \quad T(u \otimes v) \stackrel{\text{def}}{=} (2\pi h)^{-\frac{n+k}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta) (u(y) \otimes N(x,y,\theta)v) dy d\theta.$$

This operator is well-defined since  $a$  is compactly supported, but to obtain a norm estimate which is uniform in  $h$  we need to assume that  $N \in \mathcal{S}_0$ :

**Lemma 3.2.** *Suppose that  $N \in \mathcal{S}_0(\mathbb{R}^{2n+k}, \mathcal{H}, \mathcal{D})$  and that  $T$  is given by (3.14). Then*

$$(3.15) \quad \|T\|_{L^2(\mathbb{R}^n) \otimes \mathcal{H} \rightarrow L^2(\mathbb{R}^n) \otimes \mathcal{H}} = \max_{C_\psi} \frac{|a| \|N\|_{\mathcal{H} \rightarrow \mathcal{H}}}{\sqrt{|D_\psi|}} + \mathcal{O}(h),$$

where  $C_\psi \stackrel{\text{def}}{=} \{(x,y,\theta) : \partial_\theta \psi = 0\}$ , and  $D_\psi$  is given by (3.11).

If  $N \in \mathcal{S}_\delta(\mathbb{R}^{2n+k}, \mathcal{H}, \mathcal{D})$  then

$$(3.16) \quad T = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\delta m_n + \ell} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell,$$

where  $m_n$  depends only on the dimension  $n$ .

*Proof.* The estimate (3.15) follows from a standard argument based on considering  $T^*T$  and from [43, Theorem 13.13]. The estimates (3.13) with  $\delta = 0$  and  $\ell = 0$  show that the operators can be treated just as scalar symbols.

To obtain (3.16) we note that

$$\partial_{x,y,\theta}^\alpha (Q^{-L} N(x,y,\theta)) = \mathcal{O}(1) : \mathcal{H} \rightarrow \mathcal{H}, \quad \text{for } |\alpha| \delta \leq L.$$

To obtain the norm estimate (3.15) we only need a finite number of derivatives,  $m_n$ , depending only on the dimension. Taking  $L \geq m_n \delta$ , we can then apply (3.15) to the operator  $Q^{-L}T$ , which gives the bound (3.16) for  $T$ .  $\square$

A special case of is given by  $\kappa = id$ . In that case we deal with pseudodifferential operators with operator valued symbols. The following lemma summarizes their basic properties:

**Lemma 3.3.** *Suppose that  $N_j \in \mathcal{S}_{\delta_j}(\mathbb{R}^{2n})$ ,  $j = 1, 2$ . For  $u \in \mathcal{S}(\mathbb{R}^n)$  and  $v \in \mathcal{D}^\infty$  we define*

$$\text{Op}_h^w(N_j)(u \otimes v) \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y, \xi \rangle} [N_j(\frac{x+y}{2}, \xi) v] u(y) dy d\xi.$$

*These operators extend to*

$$(3.17) \quad \text{Op}_h^w(N_j) = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\ell + m_n \delta_j} \rightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell,$$

*and satisfy the following product formula:*

$$(3.18) \quad \text{Op}_h^w(N_1) \text{Op}_h^w(N_2) = \text{Op}_h^w(N_1 N_2) + hR, \quad R = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\ell + m_n(\delta_1 + \delta_2)} \rightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell.$$

*Here and in (3.17),  $\ell$  is arbitray and  $m_n$  depends only on the dimension  $n$ .*

*Proof.* When  $\delta_1 = \delta_2 = 0$  the proof is an immediate vector valued adaptation of the standard arguments presented in [43, §§4.4,4.5] where we note that only a finite number (depending on the dimension) of seminorms of symbols is needed. In general, (3.13) gives

$$(3.19) \quad \partial_{x,\xi}^\alpha Q^{-L} N_j Q^{-M} = \mathcal{O}(1) : \mathcal{D}^\ell \rightarrow \mathcal{D}^\ell, \quad |\alpha| \delta_j \leq L + M,$$

and the norm estimates (3.17) follows. To obtain the product formula we note that, using (3.19), it applies to  $Q^{-M} N_1$  and  $N_2 Q^{-M}$  for  $M$  sufficiently large depending on  $n$ . Hence

$$\begin{aligned} \text{Op}_h^w(N_1) \text{Op}_h^w(N_2) &= Q^M \text{Op}_h^w(Q^{-M} N_1) \text{Op}_h^w(N_2 Q^{-M}) Q^M \\ &= Q^M \text{Op}_h^w(Q^{-M} N_1 N_2 Q^{-M}) Q^M + Q^M \mathcal{O}(h)_{L^2 \otimes \mathcal{D}^p \rightarrow L^2 \otimes \mathcal{D}^p} Q^M \\ &= \text{Op}_h^w(N_1 N_2) + \mathcal{O}(h)_{L^2 \otimes \mathcal{D}^{p+M} \rightarrow L^2 \otimes \mathcal{D}^{p-M}}, \end{aligned}$$

which gives (3.18) provided  $m_n(\delta_1 + \delta_2) \geq 2M$ .  $\square$

We can also factorize the operator  $T$  using the pseudodifferential operators described in Lemma 3.3, the proof being an adaptation of the standard argument. When  $S : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  we also write  $S$  for  $S \otimes I_{\mathcal{H}} : L^2(\mathbb{R}^n) \otimes \mathcal{H} \rightarrow L^2(\mathbb{R}^n) \otimes \mathcal{H}$ .

**Lemma 3.4.** *Suppose that  $T$  is given by (3.14) with  $N \in \mathcal{S}_\delta$ . Then*

$$T = T^\parallel \text{Op}_h^w(N_1) + hR_1 = \text{Op}_h^w(N_2) T^\parallel + hR_2,$$

where

$$(3.20) \quad \begin{aligned} T^\parallel &\in I^{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma'_\kappa), \quad T^\parallel u(x) = (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta) u(y) dy d\theta, \\ N_2(x, d_x \psi(x,y,\theta)) &= N_1(y, -d_y \psi(x,y,\theta)) = N(x,y,\theta), \quad (x,y,\theta) \in C_\psi \\ R_j &= \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\delta m_n + \ell} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell. \end{aligned}$$

Here,  $N_j \in \mathcal{S}_\delta(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{H}, \mathcal{D})$ , and

$$\text{Op}_h^w(N_j) = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\delta m_n + \ell} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell.$$

In our applications we will have

$$(3.21) \quad \mathcal{H} = L^2(\mathbb{R}^{d_t}, d\tilde{y}), \quad Q = -\tilde{h}^2 \Delta_{\tilde{y}} + \tilde{y}^2 + 1,$$

so that  $\mathcal{D}^\ell$  are analogous to Sobolev spaces. In the rest of this section (as well in section 5), we will use the shorthand notations  $\rho_\parallel = (x, y, \theta)$  in order to shorten the expressions, and to differentiate between these variables and the “transversal variables”  $(\tilde{y}, \tilde{\eta})$ .

We consider a specific class of metaplectic operators:

$$(3.22) \quad N(\rho_\parallel) u(\tilde{y}) = (2\pi \tilde{h})^{-d_t} \int_{\mathbb{R}^{d_t}} \int_{\mathbb{R}^{d_t}} (\det \partial_{\tilde{y}, \tilde{\eta}}^2 q_{\rho_\parallel})^{\frac{1}{2}} e^{\frac{i}{\tilde{h}}(q_{\rho_\parallel}(\tilde{y}, \tilde{\eta}) - \langle \tilde{\eta}, \tilde{y}' \rangle)} u(\tilde{y}') d\tilde{y}',$$

where  $q_{\rho_\parallel}(\tilde{y}, \tilde{\eta})$  is a real quadratic form in the variables  $\tilde{y}, \tilde{\eta}$ , with coefficients depending on  $\rho_\parallel$ , being in the class  $S(\mathbb{R}^{2n+k})$ , and the matrix of coefficients  $\partial_{\tilde{y}, \tilde{\eta}}^2 q_{\rho_\parallel}$  is assumed to be

uniformly non-degenerate for all  $\rho_{\parallel}$ . The definition involves a *choice* of the branch of the square – see Remark 5.8 for further discussion of that. For any fixed  $\rho_{\parallel}$  these operators are unitary on  $\mathcal{H}$  (see for instance [43, Theorem 11.10]).

The next proposition shows that this class fits nicely into our framework:

**Proposition 3.5.** *The operators  $N(\rho_{\parallel})$  given by (3.22) satisfy*

$$(3.23) \quad \partial_{\rho_{\parallel}}^{\alpha} N(\rho_{\parallel}) = \mathcal{O}_{\alpha, \ell}(\tilde{h}^{-|\alpha|}) : \mathcal{D}^{|\alpha|+\ell} \longrightarrow \mathcal{D}^{\ell},$$

for all  $\ell$ . That means that (3.13) holds with  $\delta = 1$  (the loss in  $\tilde{h}$  is considered as dependence on  $\alpha$ ).

If  $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^{2d_t})$  is fixed,  $\Lambda > 1$ , and  $\tilde{\chi}_{\Lambda}(\bullet) \stackrel{\text{def}}{=} \tilde{\chi}(\Lambda^{-1}\bullet)$ , then for any  $\ell$  and  $k \geq 0$ ,

$$(3.24) \quad \tilde{\chi}_{\Lambda}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) = \mathcal{O}_{\ell}(\Lambda^{2k}) : \mathcal{D}^{\ell} \rightarrow \mathcal{D}^{\ell+k},$$

so that

$$(3.25) \quad \begin{aligned} \partial_{\rho_{\parallel}}^{\alpha} \left( \tilde{\chi}_{\Lambda}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) N(\rho_{\parallel}) \right) &= \mathcal{O}_{\alpha, \ell}(\Lambda^{2|\alpha|} \tilde{h}^{-|\alpha|}) : \mathcal{D}^{\ell} \longrightarrow \mathcal{D}^{\ell}, \\ \partial_{\rho_{\parallel}}^{\alpha} \left( N(\rho_{\parallel}) \tilde{\chi}_{\Lambda}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) \right) &= \mathcal{O}_{\alpha, \ell}(\Lambda^{2|\alpha|} \tilde{h}^{-|\alpha|}) : \mathcal{D}^{\ell} \longrightarrow \mathcal{D}^{\ell}. \end{aligned}$$

*Proof.* We see that  $\partial_{\rho_{\parallel}}^{\alpha} N(\rho_{\parallel})$  is an operator of the same form as (3.22) but with the amplitude multiplied by

$$\sum_{|\beta| \leq 2|\alpha|} \tilde{h}^{-m_{\beta}} \tilde{y}^{\beta_1} (\tilde{y}')^{\beta_2} \tilde{\eta}^{\beta_3} q_{\beta}(\rho_{\parallel}), \quad q_{\beta} \in S(\mathbb{R}^{2n+k}), \quad \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^{3d_t}, \quad \beta_j \in \mathbb{N}^{d_t},$$

where  $m_{\beta} \leq |\alpha|$ . Hence to obtain (3.23), it is enough to prove that

$$Q^{\ell} \tilde{y}^{\beta_1} N(\rho_{\parallel}) \left( (\tilde{y}')^{\beta_2} (\tilde{h}D_{\tilde{y}'})^{\beta_3} Q^{-\ell-|\alpha|} v(\tilde{y}') \right) = \mathcal{O}(\|v\|_{\mathcal{H}})_{\mathcal{H}}.$$

Using the exact Egorov's theorem for metaplectic operators (see for instance [43, Theorem 11.9]) we see that the left hand side is equal to

$$N(\rho_{\parallel}) \left( p_{\beta}^w(\tilde{y}', \tilde{h}D_{\tilde{y}'}) (K_q^* Q)^{\ell} Q^{-\ell-|\alpha|} v(\tilde{y}') \right), \quad K_q : (\partial_{\tilde{\eta}} q, \tilde{\eta}) \mapsto (\tilde{y}, \partial_{\tilde{y}} q),$$

$q = q_{\rho_{\parallel}}$  and where  $p_{\beta}$  is a polynomial of degree less than or equal to  $|\beta|$ . Since  $|\beta| \leq 2|\alpha|$ , the operator  $p_{\beta}^w(K_q^* Q)^{\ell} Q^{-\ell-|\alpha|}$  is bounded on  $\mathcal{H}$  (see for instance [43, Theorem 8.10]) so the unitarity of  $N$  gives the boundedness in  $\mathcal{H}$ .

To obtain (3.24) we first note that  $\tilde{\chi}_{\Lambda} \in S(\mathbb{R}^{2d_t})$  uniformly in  $\Lambda > 1$ . Hence  $Q^{-\ell} \tilde{\chi}_{\Lambda}^w Q^{\ell} = \mathcal{O}(1)_{\mathcal{H} \rightarrow \mathcal{H}}$ , uniformly in  $\Lambda$  (again, see [43, Theorem 8.10]). This gives (3.24) for  $k = 0$ . For the general case we put  $Q_{\Lambda} = 1 + \Lambda^{-2}((\tilde{h}D_{\tilde{y}})^2 + \tilde{y}^2)$ , and note that for any  $M$ ,  $Q_{\Lambda}^M \tilde{\chi}_{\Lambda}^w = \tilde{\chi}_{\Lambda, M}^w$ ,

where  $\tilde{\chi}_{\Lambda,M} \in S(\mathbb{R}^{2d_t})$  uniformly in  $\Lambda$ . Hence it is bounded on  $L^2(\mathbb{R}^{d_t})$  uniformly in  $\Lambda$  and  $\tilde{h}$ . We then write

$$\begin{aligned} Q^k \chi_{\Lambda}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) &= Q^k Q_{\Lambda}^{-k} Q_{\Lambda}^k \chi_{\Lambda}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) \\ &= (1 + (\tilde{h}D_{\tilde{y}})^2 + \tilde{y}^2)^k (1 + \Lambda^{-2}(\tilde{h}D_{\tilde{y}})^2 + \Lambda^{-2}\tilde{y}^2)^{-k} \chi_{\Lambda,k}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) \\ &= \Lambda^{2k} (1 + (\tilde{h}D_{\tilde{y}})^2 + \tilde{y}^2)^k (\Lambda^2 + (\tilde{h}D_{\tilde{y}})^2 + \tilde{y}^2)^{-k} \chi_{\Lambda,k}^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) \\ &= \mathcal{O}(\Lambda^{2k})_{L^2(\mathbb{R}^{d_t}) \rightarrow L^2(\mathbb{R}^{d_t})}, \end{aligned}$$

completing the proof of (3.24).  $\square$

#### 4. CLASSICAL DYNAMICS

In this section we will describe the consequences of the normal hyperbolicity assumption (1.16), (1.17) needed in the proof of Theorem 2.

**4.1. Stable and unstable distributions.** Let  $K^\delta$  be the trapped set (1.14) and  $E_\rho^\pm \subset T_\rho X$ ,  $\rho \in K^\delta$ , the distributions in (1.17). We recall our notation  $\varphi_t \stackrel{\text{def}}{=} \exp tH_p$  for the Hamiltonian flow generated by the function  $p(x, \xi)$ .

We start with a simple

**Lemma 4.1.** *If  $\omega$  is the canonical symplectic form on  $T^*X$  then*

$$(4.1) \quad \omega_\rho \upharpoonright_{E_\rho^\pm} = 0,$$

*that is  $E_\rho^\pm$  are isotropic.*

*Without loss of generality we can assume that the distributions  $E_\rho^\pm$  satisfy*

$$(4.2) \quad E_\rho^+ \oplus E_\rho^- = (T_\rho K^\delta)^\perp,$$

*where  $V^\perp$  denotes the symplectic orthogonal of  $V$ .*

*Proof.* The property (4.1) follows from the fact that  $\varphi_t$  preserves the symplectic structure ( $\varphi_t^* \omega = \omega$ ). for  $X, Y \in E_\rho^\pm$ ,

$$\omega_\rho(X, Y) = \omega_{\varphi_{\mp t}(\rho)}((d\varphi_{\mp t})(\rho)X, d\varphi_{\mp t}(\rho)Y) \rightarrow 0, \quad t \rightarrow +\infty$$

To see that we can assume (4.2) we note that the distribution  $\{(T_\rho K^\delta)^\perp, \rho \in K^\delta\}$  is invariant by the flow:  $d\varphi_t(\rho) : T_\rho K \rightarrow T_{\varphi_t(\rho)} K$ , and  $d\varphi_t(\rho)$  is a symplectic transformation. If  $\pi_\rho : T_\rho(T^*X) \rightarrow (T_\rho K^\delta)^\perp$  is the symplectic projection, then  $\pi_{d\varphi_t(\rho)} \circ d\varphi_t(\rho) = d\varphi_t(\rho) \circ \pi_\rho$ . This means that we may safely replace  $E_\rho^\pm$  with  $\pi_\rho(E_\rho^\pm)$ , without altering the properties (1.17).  $\square$

**4.2. Construction of the escape function.** To construct the escape function near the trapped set we need a lemma concerning invariant cones near  $K^\delta$ . To define them we introduce a Riemannian metric on  $T^*X$  and use the tubular neighbourhood theorem (see for instance [27, Appendix C.5]) to make the identifications

$$\begin{aligned}
 \text{neigh}(K^\delta) &\simeq N^*K^\delta \cap \{(\rho, \zeta) \in T^*(T^*X) : \|\zeta\|_\rho \leq \epsilon_1\} \\
 (4.3) \quad &\simeq (TK^\delta)^\perp \cap \{(\rho, z) \in T(T^*X) : \|z\|_\rho \leq \epsilon_1\} \\
 &\simeq \{(m, z) : m \in K^\delta, z \in \mathbb{R}^{2d_t}, \|z\|_\rho \leq \epsilon_1\}.
 \end{aligned}$$

Here  $(TK^\delta)^\perp$  denotes the symplectic orthogonal of  $TK^\delta \subset T_{K^\delta}(T^*X) \subset T(T^*X)$ . Since  $K^\delta$  is symplectic, the symplectic form identifies  $(TK^\delta)^\perp$  with the conormal bundle  $N^*K^\delta$ . The norm  $\|\bullet\|_\rho$  is a smoothly varying norm on  $T_\rho(T^*X)$ . We write  $d_\rho(z, z') = \|z - z'\|_\rho$  and introduce a distance function  $d : \text{neigh}(K^\delta) \times \text{neigh}(K^\delta) \rightarrow [0, \infty)$  obtained by choosing a Riemannian metric on  $\text{neigh}(K^\delta)$ . We have  $d((m, z), (m, z')) \sim d_m(z, z')$  and the notation  $a \sim b$ , here and below, means that there exists a constant  $C \geq 1$  (independent of other parameters) such that  $b/C \leq a \leq Cb$ .

Assuming that  $E_\rho^\pm$  are chosen so that (4.2) holds we can define (closed) cone fields by putting

$$\begin{aligned}
 (4.4) \quad C_\rho^\pm &\stackrel{\text{def}}{=} \{z \in (T_\rho K^\delta)^\perp : d_\rho(z, E_\pm^\rho) \leq \epsilon_2 \|z\|_\rho, \|z\|_\rho \leq \epsilon_1\}, \\
 C^\pm &\stackrel{\text{def}}{=} \bigcup_{\rho \in K^\delta} C_\rho^\pm \subset \text{neigh}(K^\delta),
 \end{aligned}$$

where we used the identification (4.3). Since the maps  $\rho \mapsto E_\rho^\pm$  are continuous,  $C^\pm$  are closed.

The basic properties  $C^\pm$  are given in the following

**Lemma 4.2.** *There exists  $t_0 > 0$  and  $\epsilon_2^0 > 0$  such that, for every  $t > t_0$  there exists  $\epsilon_1^0$  such that if one chooses  $\epsilon_j < \epsilon_j^0$ ,  $j = 1, 2$  in the definition of  $\text{neigh}(K^\delta)$  and  $C^\pm$ , then*

$$(4.5) \quad \rho \in C_\pm, \varphi_{\pm t}(\rho) \in \text{neigh}(K^\delta) \implies \varphi_{\pm t}(\rho) \in C_\pm.$$

*In fact a stronger statement is true: for some constant  $\lambda_1 > 0$  and any  $t \geq t_0$ ,*

$$(4.6) \quad \rho, \varphi_{\pm t}(\rho) \in \text{neigh}(K^\delta) \implies d(\varphi_{\pm t}(\rho), C^\pm) \leq e^{-\lambda_1 t} d(\rho, C^\pm).$$

*Finally,*

$$(4.7) \quad d(\rho, C^+)^2 + d(\rho, C^-)^2 \sim d(\rho, K^\delta)^2.$$

The conclusions (4.6) and (4.7) are similar to [31, Lemma 4.3] and [34, Lemma 5.2] but the proof does not use foliations by stable and unstable manifolds which seem different under our assumptions.

*Proof.* For  $\rho \in \text{neigh}(K^\delta)$  let  $(m, z)$ ,  $m \in K^\delta$  and  $z \in \mathbb{R}^{2d_t} \simeq (T_m K^\delta)^\perp$  be local coordinates near  $\rho$ . Similarly let  $(\tilde{m}, \tilde{z})$  be local coordinates near  $\varphi_t(\rho) \in \text{neigh}(K^\delta)$  (by assumption in (4.5)). Then if for each  $m$  we put  $d_\perp \varphi_t(m) \stackrel{\text{def}}{=} d\varphi_t(m)|_{(T_m K^\delta)^\perp}$ , the map  $\varphi_t$  can be written as,

$$(4.8) \quad \begin{aligned} \varphi_t(m, z) &= (\varphi_t(m) + \mathcal{O}_t(\|z\|^2), d_\perp \varphi_t(m)z + \mathcal{O}_t(\|z\|^2)) \\ &= (\varphi_t(m_1), d_\perp \varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2)), \quad m_1 = m + \mathcal{O}_t(\|z\|^2). \end{aligned}$$

(Here we identify  $(T_m K^\delta)^\perp$  with  $\mathbb{R}^{2d_t}$  and consider  $d_\perp \varphi_t(m) : \mathbb{R}^{2d_t} \rightarrow \mathbb{R}^{2d_t}$ , with similar identification near  $\varphi_t(\rho)$ . The norm  $\|\bullet\|$  is now fixed in that neighbourhood.)

Let  $z = z_+ + z_-$  be the decomposition of  $z$  corresponding to  $(T_{m_1} K^\delta)^\perp = E_{m_1}^+ \oplus E_{m_1}^-$  (we assumed without loss of generality that (4.2) holds). The continuity of  $\rho \mapsto E_\rho^\pm$  and the definition of  $C_\rho^+$  show that if  $\epsilon_1$  is small enough depending on  $t$  (so that  $d(m, m_1) = \mathcal{O}_t(\|z\|^2)$  is small)

$$(4.9) \quad z \in C_m^+ \implies \|z_-\|_{m_1} \leq 2\epsilon_2 \|z_+\|_{m_1}.$$

Since

$$d_\perp \varphi_t(m_1)z = \sum_{\pm} d_\perp \varphi_t(m_1)z_\pm, \quad d_\perp \varphi_t(m_1)z_\pm \in E_{\varphi_t(m_1)}^\pm,$$

and, by normal hyperbolicity, for some  $C > 0$  and  $\lambda_1 > 0$  one has for all positive times

$$(4.10) \quad \begin{aligned} \|d_\perp \varphi_t(m_1)z_+\| &\geq \frac{1}{C} e^{2\lambda_1 t} \|z_+\|, \\ \|d_\perp \varphi_t(m_1)z_-\| &\leq C e^{-2\lambda_1 t} \|z_-\|. \end{aligned}$$

If  $z \in C_m^+$ , then this and (4.9) show

$$\|d_\perp \varphi_t(m_1)z_-\| \leq 2C^2 e^{-4\lambda_1 t} \epsilon_2 \|d_\perp \varphi_t(m_1)z_+\|.$$

Let us take  $t_0$  such that  $2C^2 e^{-4\lambda_1 t_0} < 1/2$ . For  $t \geq t_0$  and  $\epsilon_1$  small enough depending on  $t$  this shows that

$$(4.11) \quad z \in C_m^+ \text{ and } \|z\| \leq \epsilon_1, \quad \|d_\perp \varphi_t(m_1)z\| \leq \epsilon_1 \implies d_\perp \varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2) \in C_{\varphi_t(m_1)}^+,$$

which in view of (4.8) proves (4.5) in the  $+$  case with the  $-$  case being essentially the same.

To obtain (4.6) we note that for  $(m, z) \in \text{neigh}(\rho, K^\delta)$ ,

$$d((m, z), C^+) \sim d_m(z, C_m^+) \sim \|z_-\|(1 - \mathbb{1}_{C_m^+}(z)), \quad z = z_+ + z_-, \quad z_\pm \in E_m^\pm,$$

where  $\mathbb{1}_A$  is the characteristic function of a set  $A$ . (To see the first  $\sim$  we need to show that  $d((m, z), C^+) \leq c_0 d_m(z, C_m^+)$  for some  $c_0$ , which follows from an argument by contradiction using pre-compactness of  $K^\delta$ .)

We also observe that if  $d\varphi_t(m_1)z \in \text{neigh}(K^\delta)$  then (4.11) gives, for  $\epsilon_1$  small enough depending on  $t$ ,

$$1 - \mathbb{1}_{C_{\varphi_t(m_1)}^+}(d_\perp \varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2)) \leq 1 - \mathbb{1}_{C_m^+}(z).$$

Hence, using (4.8) and (4.10), writing  $z = z_- + z_+$  as before, and taking  $\epsilon_1$  sufficiently small depending on  $t \geq t_0$ ,

$$\begin{aligned}
d(\varphi_t(m, z), C^+) &\sim d_{\varphi_t(m_1)}(d_\perp \varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2), C_{\varphi_t(m_1)}^+) \\
&\sim \|d_\perp \varphi_t(m_1)z_-\| (1 + \mathcal{O}_t(\|z_-\|)) (1 - \mathbb{1}_{C_{\varphi_t(m_1)}^+}(d_\perp \varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2))) \\
&\leq C e^{-2\lambda_1 t} \|z_-\| (1 + \mathcal{O}_t(\|z_-\|)) (1 - \mathbb{1}_{C_{m_1}^+}(z)) \\
&\leq C' e^{-2\lambda_1 t} d_{m_1}(z, C_{m_1}^+) \sim C' e^{-2\lambda_1 t} d_m(z, C_m^+) \\
&\leq e^{-\lambda_1 t} d((m, z), C^+).
\end{aligned}$$

Here in the second line we used the fact that  $\|z\| \leq C\|z_-\|$  if the distance is non zero (with  $C$  depending on  $\epsilon_2$ ). In the fourth line we used the continuity of the cone field,  $m \mapsto C_m^+$ .

This proves (4.6). The last claim (4.7) is immediate from the construction of  $C^\pm$  and the fact that  $E_\rho^+ \cap E_\rho^- = \{0\}$ .  $\square$

We now regularize  $d(\rho, C^\pm)^2$  uniformly with respect to a parameter  $\epsilon$ . It will eventually be taken to be  $h/\tilde{h}$ , where  $\tilde{h}$  is a small constant independent of  $h$ . Lemma 4.2 and the arguments of [31, §4] and [35, §7] immediately give

**Lemma 4.3.** *There exists  $t_0 > 0$  such that for any  $t > t_0$ , there exists a neighbourhood  $\mathcal{V}_t$  of  $K^{2\delta}$  and a constant  $C_0 > 0$  such that the following holds.*

*For any small  $\epsilon > 0$  there exist functions  $\gamma_\pm \in \mathcal{C}^\infty(\mathcal{V}_t \cup \varphi_t(\mathcal{V}_t))$  such that for  $\rho \in \mathcal{V}_t \cap p^{-1}([-\delta, \delta])$ ,*

$$\begin{aligned}
(4.12) \quad &\gamma_\pm(\rho) \sim d(\rho, C_\pm)^2 + \epsilon, \quad \gamma_\pm(\rho) \geq \epsilon, \\
&\pm(\gamma_\pm(\rho) - \gamma_\pm(\varphi_t(\rho))) + C_0\epsilon \sim \gamma_\pm(\rho), \\
&\partial^\alpha \gamma_\pm(\rho) = \mathcal{O}(\gamma_\pm(\rho)^{1-|\alpha|/2}), \\
&\gamma_+(\rho) + \gamma_-(\rho) \sim d(\rho, K^\delta)^2 + \epsilon.
\end{aligned}$$

Following [31, §4] and [35, §7] again this gives us an escape function for a small neighbourhood of the trapped set. We record this in

**Proposition 4.4.** *Let  $\gamma_\pm$  be the functions given in Lemma 4.3. For some  $M \gg 1$  independent of  $\epsilon$ , define*

$$(4.13) \quad G_0 \stackrel{\text{def}}{=} \log(M\epsilon + \gamma_-) - \log(M\epsilon + \gamma_+)$$

*on a neighbourhood  $\mathcal{V}$  of the trapped set  $K^{2\delta}$ .*

*For any  $t_0$  large enough we can find a neighbourhood of  $U_1 \subseteq \mathcal{V}$  of  $K^{2\delta}$  and  $C_1 > 0$ , such that*

$$G_0 = \mathcal{O}(\log(1/\epsilon)), \quad \partial_\rho^\alpha G_0 = \mathcal{O}(\min(\gamma_+, \gamma_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \geq 1,$$

and such that for  $\rho \in U_1 \cap p^{-1}([-\delta, \delta])$ ,

$$(4.14) \quad \begin{aligned} \partial_\rho^\alpha (G_0(\varphi_{t_0}(\rho)) - G_0(\rho)) &= \mathcal{O}(\min(\gamma_+, \gamma_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \geq 0, \\ d(\rho, K^\delta)^2 \geq C_1 \epsilon &\implies G_0(\varphi_{t_0}(\rho)) - G_0(\rho) \geq 1/C_1. \end{aligned}$$

In the intermediate region between  $U_1$  and  $\{x : w(x) > 0\}$  we need an escape function similar to the one constructed in [14, §4] and [24, Appendix]. We work here under the general assumptions of §1.3 and present a slightly modified argument.

**Lemma 4.5.** *Suppose that  $X$  is a compact smooth manifold,  $p \in S^m(T^*X; \mathbb{R})$ ,  $w \in S^k(T^*X; [0, \infty))$ ,  $k \leq m$ , and that (1.9) holds. For any open neighbourhood  $V_1$  of  $K^{3\delta}$ , there exists  $\epsilon_0 > 0$  and a function  $G_1 \in \mathcal{C}_c^\infty(p^{-1}((-2\delta, 2\delta)))$  such that*

$$(4.15) \quad \begin{aligned} G(\rho) &= 0 \quad \text{for } \rho \text{ in some neighbourhood of } K^{3\delta}, \\ H_p G_1(\rho) &\geq 0 \quad \text{for } \rho \notin w^{-1}((\epsilon_0, \infty)), \\ H_p G_1(\rho) &> 0 \quad \text{for } \rho \in p^{-1}([-\delta, \delta]) \setminus (V_1 \cup w^{-1}((\epsilon_0, \infty))). \end{aligned}$$

*Proof.* Call  $U_0 \stackrel{\text{def}}{=} w^{-1}((0, \infty))$  and suppose  $\rho \in p^{-1}([-2\delta, 2\delta]) \setminus (V_1 \cup U_0)$ . We first claim that there exist  $T_\pm = T_\pm(\rho)$ ,  $T_- < 0 < T_+$ , such that

$$(4.16) \quad \varphi_{T_+}(\rho) \in U_0 \quad \text{or} \quad \varphi_{T_-}(\rho) \in U_0,$$

$$(4.17) \quad \varphi_{T_\pm}(\rho) \in V_1 \cup U_0.$$

(Here and below we use the notation  $\varphi_A(\rho) = \{\varphi_t(\rho) : t \in A\}$ .)

To justify these claims we first note that since  $\rho \notin K^{2\delta}$ ,  $\varphi_{\mathbb{R}}(\rho) \cap U_0 \neq \emptyset$  which implies that

$$(4.18) \quad \exists T_1, \quad \varphi_{T_1}(\rho) \in U_0.$$

Assuming that  $T_1 < 0$  we want to show that  $\varphi_{T_2}(\rho) \in V_1 \cup U_0$  for some  $T_2 > 0$ . Suppose that this is not true, that is

$$(4.19) \quad \varphi_{(0, \infty)}(\rho) \cap (V_1 \cup U_0) = \emptyset.$$

Then for any  $t_j \rightarrow \infty$ ,

$$\rho_j \stackrel{\text{def}}{=} \varphi_{t_j}(\rho) \in p^{-1}([-2\delta, 2\delta]) \setminus (V_1 \cup U_0), \quad \varphi_{[0, \infty)}(\rho_j) \cap (V_1 \cup U_0) = \emptyset.$$

By (1.9) the set  $p^{-1}([-2\delta, 2\delta]) \setminus (V_1 \cup U_0)$  is compact and hence, by passing to a subsequence, we can assume that  $\rho_j \rightarrow \bar{\rho} \notin V_1 \cup U_0$ . We have  $\varphi_t(\rho_j) \rightarrow \varphi_t(\bar{\rho})$ , as  $j \rightarrow \infty$ , uniformly for  $|t| \leq T$ , and it follows that  $\varphi_{[0, \infty)}(\bar{\rho}) \cap (V_1 \cup U_0) = \emptyset$ . For  $t \geq -t_j$ ,

$$\varphi_t(\rho_j) = \varphi_{t+t_j}(\rho) \subset \varphi_{[0, \infty)}(\rho) \subset p^{-1}([-2\delta, 2\delta]) \setminus (V_1 \cup U_0),$$

which means that  $\varphi_t(\bar{\rho}) \notin V_1 \cup U_0$  for  $t > -t_j \rightarrow -\infty$ . We conclude that

$$\varphi_{\mathbb{R}}(\bar{\rho}) \cap V_1 \cup U_0 = \emptyset \implies \varphi_{\mathbb{R}}(\bar{\rho}) \in K^{3\delta}.$$



This contradicts the property  $\bar{\rho} \notin V_1$ , and proves the existence of  $T_2 > 0$  such that  $\varphi_{T_2}(\rho) \in V_1 \cup U_0$ . We call  $T_-(\rho) = T_1$ ,  $T_+(\rho) = T_2$ .

In the case  $T_1$  in (4.18) is positive, a similar argument shows the existence of  $T_2 < 0$  such that  $\varphi_{T_2}(\rho) \in (V_1 \cup U_0) \neq \emptyset$ . In this case we call  $T_-(\rho) = T_2$ ,  $T_+(\rho) = T_1$ .

For each  $\rho \in p^{-1}([-2\delta, 2\delta])$  we can find an open hypersurface  $\Gamma_\rho$ , transversal to  $H_p$  at  $\rho$ , such that, if  $\varphi_{T_\pm}(\rho) \in U_0$ , then for  $\rho' \in \Gamma_\rho$ ,

$$\varphi_{T_\pm}(\rho') \in U_0, \quad \varphi_{T_\mp}(\rho') \in V_1 \cup U_0.$$

Notice that the closure of the tube  $\Omega_\rho \stackrel{\text{def}}{=} \varphi_{(T_-, T_+)}(\Gamma_\rho)$  does not intersect  $K^{3\delta}$ . Using this tube, we construct a local escape functions  $g_\rho \in \mathcal{C}_c^\infty(\Omega_\rho)$ , with the following properties: for some  $\epsilon_\rho > 0$ , and an slightly smaller tube  $\Omega'_\rho \subset \Omega_\rho$  containing  $\varphi_{(T_-, T_+)}(\rho)$ ,

$$(4.20) \quad H_p g_\rho(\rho') \geq 0, \quad \rho' \notin w^{-1}((\epsilon_\rho, \infty)), \quad H_p g_\rho(\rho') > 0, \quad \rho' \in \Omega'_\rho \setminus (w^{-1}((\epsilon_\rho, \infty)) \cup V_1).$$

Here  $\epsilon_\rho$  is chosen so that if  $\varphi_{T_\pm}(\rho) \in U_0$  then  $\varphi_{T_\pm}(\Gamma_\rho) \subset w^{-1}((2\epsilon_\rho, \infty))$ .

To construct  $g_\rho$  we take  $(t, m) \in (T_-, T_+) \times \Gamma_\rho$  as local coordinates:  $(t, m) \mapsto \varphi_t(m) \in \Omega_\rho$ . Suppose that  $\varphi_{T_-}(\rho) \in U_0$ , and that  $\varphi_{(T_-, T_- + \gamma)}(\Gamma_\rho) \subset w^{-1}((\epsilon_\rho, \infty))$  and  $\varphi_{(T_+ - \gamma, T_+)}(\Gamma_\rho) \subset V_1 \cup U_0$ . Choose  $\chi_\rho \in \mathcal{C}_c^\infty((T_-, T_+))$  which is strictly increasing on  $(T_- + \gamma, T_+ - \gamma)$  and non-decreasing on  $(T_+ - \gamma, T_+)$ . Also, choose  $\psi_\rho \in \mathcal{C}_c^\infty(\Gamma_\rho)$  with  $\psi_\rho(\rho) = 1$ . Then put  $g_\rho(\varphi_t(m)) \stackrel{\text{def}}{=} \chi_\rho(t)\psi_\rho(m)$ . Since  $H_p g_\rho = \chi'_\rho(t)\psi_\rho(m)$ , (4.20) holds. A similar construction can be applied in the case where  $\varphi_{T_-}(\rho) \in V_1$ ,  $\varphi_{T_+}(\rho) \in U_0$ .

From the open cover

$$p^{-1}([- \delta, \delta]) \setminus (V_1 \cup U_0) \subset \bigcup \{ \Omega_\rho : \rho \in p^{-1}([- \delta, \delta]) \setminus (V_1 \cup U_0) \},$$

one may extract a finite subcover  $\bigcup_{j=1}^L \Omega_{\rho_j}$ . The closure of this cover does not intersect  $K^{3\delta}$ , so that the function  $G_1(\rho) \stackrel{\text{def}}{=} \sum_{l=1}^L g_{\rho_l}(\rho)$  satisfies (4.15), for  $\epsilon_0 = \min_j \epsilon_{\rho_j}$ .  $\square$

We conclude this section with a global escape function which combines the ones in Proposition 4.4 and Lemma 4.5. The estimates will be needed to justify the quantization of the escape function in §6. The proof is an immediate adaptation of the proof of [31, Proposition 4.6] and is omitted.

**Proposition 4.6.** *Let  $\mathcal{V}$ ,  $U_1$ ,  $G_0$  and  $t_0$  be as in Proposition 4.4, and let  $W_1$  be a neighbourhood of  $K^{2\delta}$  such that  $W_1 \Subset U_1$ ,  $W_1 \cup \varphi_{t_0}(W_1) \Subset \mathcal{V}$ .*

*Take  $\chi \in \mathcal{C}_c^\infty(\mathcal{V})$  equal to 1 in  $W_1 \cup \varphi_{t_0}(W_1)$ , and let  $G_1$  be the escape function constructed in Lemma 4.5 for  $V_1 = W_1$ . Then for any  $\Gamma > 1$ ,  $G \in \mathcal{C}_c^\infty(T^*X; \mathbb{R})$  defined by*

$$(4.21) \quad G \stackrel{\text{def}}{=} \chi C_3 \Gamma G_0 + C_4 \log(1/\epsilon) G_1$$

where  $C_3$  and  $C_4$  are sufficiently large, satisfies the following estimates

$$\begin{aligned}
 (4.22) \quad & |G(\rho)| \leq C_6 \log(1/\epsilon), \quad \partial^\alpha G = \mathcal{O}(\epsilon^{-|\alpha|/2}), \quad |\alpha| \geq 1, \\
 & \rho \in W_1 \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq -C_7, \\
 & \rho \in W_1 \cap p^{-1}([-\delta, \delta]), \quad d(\rho, K^\delta)^2 \geq C_1 \epsilon \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq 2\Gamma, \\
 & \rho \in p^{-1}([-\delta, \delta]) \setminus (W_1 \cup w^{-1}((\epsilon_0, \infty))) \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq C_8 \log(1/\epsilon).
 \end{aligned}$$

In addition we have

$$(4.23) \quad \frac{\exp G(\rho)}{\exp G(\rho')} \leq C_9 \left(1 + \frac{d(\rho, \rho')}{\sqrt{\epsilon}}\right)^{N_1},$$

for some constants  $C_9$  and  $N_1$ .

## 5. ANALYSIS NEAR THE TRAPPED SET

In this section we will analyse the cut-off propagator

$$(5.1) \quad \chi^w \exp(-itP/h) \chi^w,$$

where  $\chi^w = \text{Op}_h^w(\chi)$ ,  $\chi \in \mathcal{C}_c^\infty \cap \tilde{S}_{\frac{1}{2}}$  and  $\text{supp } \chi \subset \{\rho : d(\rho, K^\delta) < R(h/\tilde{h})^{\frac{1}{2}}\}$  for some  $R > 1$  independent of  $\tilde{h}$ ,  $h$ . We could take two different cut-offs on both sides, as long as they share the above properties.

Our objective is to prove the following bound (announced in (2.4)):

**Proposition 5.1.** *For any  $\epsilon > 0$  and  $M > 0$ , there exist  $C_0 > 0$ ,  $\tilde{h}_0$ , and a function  $\tilde{h} \mapsto h_0(\tilde{h}) > 0$ , such that for  $0 < \tilde{h} < \tilde{h}_0$  and  $0 < h < h_0(\tilde{h})$ ,*

$$(5.2) \quad \|\chi^w e^{-itP/h} \chi^w\|_{L^2 \rightarrow L^2} \leq C_0 \tilde{h}^{-d_t/2} \exp\left(-\frac{1}{2}t(\lambda_0 - \epsilon)\right), \quad 0 \leq t \leq M \log 1/\tilde{h},$$

where  $\lambda_0$  is given by (1.20).

Since  $e^{-itP/h}$  is unitary, the above bound is nontrivial only for  $0 \leq t \leq \frac{d_t}{\lambda_0} \log 1/\tilde{h} \leq t \leq M \log 1/\tilde{h}$ .

**5.1. Darboux coordinate charts.** We start by setting up an adapted atlas of Darboux coordinate charts near  $K^\delta$ , that is take a finite open cover

$$K^\delta \subset \bigcup_{j \in J} U_j,$$

and symplectomorphisms  $\kappa_j : U_j \rightarrow V_j = \text{neigh}(0, \mathbb{R}^{2d})$ . The standard symplectic coordinates on  $V_j$  then appear as a local symplectic coordinate frame on  $U_j$ . We may choose the coordinates such that they split into

$$X = (x, y), \quad \Xi = (\xi, \eta), \quad y, \eta \in \mathbb{R}^{d_t}, \quad x, \xi \in \mathbb{R}^{d-d_t},$$

such that the symplectic submanifold  $K^\delta \cap U_j$  is identified with  $\mathcal{K} \cap V_j \subset \mathbb{R}^{2d}$ , where

$$\mathcal{K} \stackrel{\text{def}}{=} \{y = \eta = 0\} \subset \mathbb{R}^{2d}.$$

That is,  $(x, \xi)$  is a local coordinate frame on  $K^\delta$ , while  $(y, \eta)$  describes the transversal directions.

We also assume that for each  $\rho \in K^\delta \cap U_j$ , identified with some  $(x, \xi, 0, 0) \in V_j$ , the subspace  $\{(x, y, \xi, 0), y \in \mathbb{R}^{d_t}\}$  is  $\epsilon$ -close to the transversal unstable space  $d\kappa_j(E_\rho^+)$ , while the subspace  $\{(x_0, 0, \xi_0, \eta), \eta \in \mathbb{R}^{d_t}\}$  is  $\epsilon$ -close to the transversal stable space  $d\kappa_j(E_\rho^-)$ .

We want to describe the flow in the vicinity of  $K^\delta$ , using these local coordinates. We choose a (large) time  $t_0 > 0$ , and express the time- $t_0$  flow  $\varphi_{t_0} : U_{j_0} \rightarrow U_{j_1}$  in the local coordinate frames, through the maps

$$(5.3) \quad \kappa_{j_1 j_0} \stackrel{\text{def}}{=} \kappa_{j_1} \circ \varphi_{t_0} \circ \kappa_{j_0}^{-1} : D_{j_1 j_0} \rightarrow A_{j_1 j_0},$$

where  $D_{j_1 j_0} \subset V_{j_0}$  is the *departure set*, while  $A_{j_1 j_0} \subset V_{j_1}$  is the *arrival set*. This is defined when  $\varphi_{t_0}(U_{j_0}) \cap U_{j_1} \neq \emptyset$  and such a pair  $j_1 j_0$  for which this holds will be called *physical*.

Below we will also consider the maps  $\kappa_{j_1 j_0}^n$  representing the time- $nt_0$  flow in the charts  $V_{j_0} \rightarrow V_{j_1}$ .

**5.2. Splitting  $e^{-iP/h}$  into pieces.** We want to use the fact that the propagator  $e^{-it_0 P/h}$  is a Fourier integral operator on  $M$  associated with  $\varphi_{t_0}$ . To make this remark precise, we will use a smooth partition of unity  $(\pi_j \in \mathcal{C}_c^\infty(U_j, [0, 1]))$  such that each cut-off  $\pi_j$  is equal to unity near some  $\tilde{U}_j \Subset U_j$ , and the quantized cut-offs  $\Pi_i \stackrel{\text{def}}{=} \text{Op}_h^w(\pi_i)$  satisfy the following quantum partition of unity:

$$(5.4) \quad \Pi \stackrel{\text{def}}{=} \sum_{j=1}^J \Pi_j \Pi_j^* \equiv I \quad \text{microlocally in a neighbourhood of } K^\delta.$$

We will then split  $e^{-it_0 P/h}$  into the local propagators

$$(5.5) \quad T_{j_1 j_0}^b \stackrel{\text{def}}{=} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0},$$

which can be represented by operators on  $L^2(\mathbb{R}^d)$  as follows. We define Fourier integral operators  $\mathcal{U}_j : L^2(X) \rightarrow L^2(\mathbb{R}^d)$  quantizing the coordinate changes  $\kappa_j$ , and microlocally unitary in some subset of  $V_j \times U_j$  containing  $\kappa_j(\text{supp } \pi_j) \times \text{supp } \pi_j$ , so that

$$(5.6) \quad \forall j, \quad \Pi_j \Pi_j^* = \Pi_j \mathcal{U}_j^* \mathcal{U}_j \Pi_j^* + \mathcal{O}(h^\infty),$$

The local propagators  $T_{j_1 j_0}^b$  are then represented by

$$(5.7) \quad T_{j_1 j_0} \stackrel{\text{def}}{=} \mathcal{U}_{j_1} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0} \mathcal{U}_{j_0}^*.$$

Notice that for an unphysical pair  $j_1 j_0$ ,  $T_{j_1 j_0} = \mathcal{O}(h^\infty)_{L_2 \rightarrow L_2}$ . For a physical pair  $j_1 j_0$ ,  $T_{j_1 j_0}$  is an Fourier integral operator associated with the local symplectomorphism  $\kappa_{j_1 j_0}$ .

From the unitarity of  $e^{-it_0 P/h}$  we draw the following property of the operators  $T_{j'j}$ .

**Lemma 5.2.** *The operator-valued matrix  $\mathbf{T} \stackrel{\text{def}}{=} (T_{ij})_{i,j=1,\dots,J}$ , acting on the space  $L^2(\mathbb{R}^d)^J$  with the Hilbert norm  $\|\mathbf{u}\|^2 = \sum_{j=1}^J \|u_j\|_{L^2}^2$ , satisfies*

$$\|\mathbf{T}\|_{L^2(\mathbb{R}^d)^J \rightarrow L^2(\mathbb{R}^d)^J} = 1 + \mathcal{O}(h).$$

*Proof.* From (5.6), the action of  $T_{j_1 j_0}$  on  $L^2(\mathbb{R}^d)$  is (up to an error  $\mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}$ ) unitarily equivalent with the action of  $T_{j_1 j_0}^\flat$  on  $L^2(X)$ . Hence, the action of  $\mathbf{T}$  on  $L^2(\mathbb{R}^d)^J$  is equivalent to the action of  $\mathbf{T}^\flat$  on  $L^2(X)^J$ , where  $\mathbf{T}^\flat$  is the matrix of operators (5.5).

To obtain the norm estimate we follow [2, Lemma 6.5], put  $\mathcal{H} \stackrel{\text{def}}{=} L^2(X)$ ,  $U = e^{-it_0 P/h}$ , and define the row vector of cut-off operators  $C = (\Pi_i)_{i=1,\dots,J}$ . The operator valued matrix  $\mathbf{T}^\flat$  can be written as  $\mathbf{T}^\flat = C^*(U \otimes I_J)C$ . Its operator norm on  $\mathcal{L}(\mathcal{H}^J)$  satisfies

$$\begin{aligned} \|\mathbf{T}^\flat\|_{\mathcal{L}(\mathcal{H}^J)}^2 &= \|(\mathbf{T}^\flat)^* \mathbf{T}^\flat\|_{\mathcal{L}(\mathcal{H}^J)} = \|C^*(U \otimes I_J)C C^*(U^* \otimes I_J)C\|_{\mathcal{L}(\mathcal{H}^J)} \\ &= \|C^*(U \Pi U^* \otimes I_J)C\|_{\mathcal{L}(\mathcal{H}^J)}. \end{aligned}$$

Egorov's theorem (see (3.6)) and [43, Theorem 13.13] imply that  $\Pi^1 \stackrel{\text{def}}{=} U \Pi U^*$  is a positive semidefinite operator of norm  $1 + \mathcal{O}(h)$ , with symbol equal to  $1 + \mathcal{O}(h)$  near  $K^\delta$ , and its square root  $\sqrt{\Pi^1}$ , as well as the product  $\sqrt{\Pi^1} \Pi \sqrt{\Pi^1}$ , have the same properties. Hence,

$$\begin{aligned} \|\mathbf{T}^\flat\|_{\mathcal{L}(\mathcal{H}^J)}^2 &= \|((\sqrt{\Pi^1} \otimes I_J) C)^* (\sqrt{\Pi^1} \otimes I_J) C\|_{\mathcal{L}(\mathcal{H}^J)} \\ &= \|(\sqrt{\Pi^1} \otimes I_J) C ((\sqrt{\Pi^1} \otimes I_J) C)^*\|_{\mathcal{L}(\mathcal{H})} \\ &= \|\sqrt{\Pi^1} \Pi \sqrt{\Pi^1}\|_{\mathcal{L}(\mathcal{H})} = 1 + \mathcal{O}(h). \end{aligned} \quad \square$$

**5.3. Iterated propagator.** In this section we explain how to use the  $T_{j'j}$  to study our cut-off propagator (5.1).

First of all, Egorov's theorem (3.7) applied to  $T = \Pi_j \mathcal{U}_j^*$ ,  $B_2 = \chi^w$  allows us to write

$$(5.8) \quad \mathcal{U}_j \Pi_j^* \chi^w = \chi_j^w \mathcal{U}_j \Pi_j^* + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \rightarrow L^2}, \quad j = 1, \dots, J,$$

where the symbol  $\chi_j = \chi \circ \kappa_j^{-1} \in \tilde{S}_{\frac{1}{2}}(T^* \mathbb{R}^d)$ .

We start from a arbitrary normalized state  $u \in L^2(X)$ , and represent the part of  $u$  microlocalized near  $K^\delta$  through the (column) vector of states

$$\mathbf{u} \stackrel{\text{def}}{=} (u_j)_{j=1,\dots,J}, \quad u_j \stackrel{\text{def}}{=} \mathcal{U}_j \Pi_j^* u, \quad \|\mathbf{u}\|^2 \stackrel{\text{def}}{=} \sum_j \|u_j\|^2 = \langle u, \Pi u \rangle + \mathcal{O}(h^\infty) \|u\|_{L^2}^2.$$

The equations (5.7) and (5.8) show that

$$\begin{aligned}
\Pi e^{-it_0 P/h} \chi^w u &= \sum_j \Pi_j \Pi_j^* e^{-it_0 P/h} \chi^w u = \sum_{j_1, j_0} \Pi_{j_1} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0} \Pi_{j_0}^* \chi^w u \\
&= \sum_{j_1, j_0} \Pi_j \mathcal{U}_{j_1}^* \mathcal{U}_{j_1} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0} \mathcal{U}_{j_0}^* \mathcal{U}_{j_0} \Pi_{j_0}^* \chi^w u + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2} \\
&= \sum_{j_1, j_0} \Pi_{j_1} \mathcal{U}_{j_1}^* T_{j_1 j_0} \chi_{j_0}^w u_{j_0} + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \rightarrow L^2}.
\end{aligned}$$

Similarly, for  $n \geq 2$  the propagator  $e^{-int_0 P/h}$  can be represented by iteratively applying the operator valued matrix  $\mathbf{T}$  to the vector  $\mathbf{u}$ . By inserting the identities (5.4), (5.6)  $n$  times in the expression  $\chi^w e^{-int_0 P/h} \chi^w u$ , we get the following

**Lemma 5.3.** *For any  $n \in \mathbb{N}$  (independent of  $h$ ), we have*

$$\begin{aligned}
\Pi e^{-int_0 P/h} \chi^w u &= \sum_{j_n, \dots, j_0} \Pi_{j_n} \mathcal{U}_{j_n}^* T_{j_n j_{n-1}} \cdots T_{j_1 j_0} \chi_{j_0}^w u_{j_0} + \mathcal{O}_n(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \rightarrow L^2} \\
(5.9) \quad &= \sum_{j_n, j_0} \Pi_{j_n} \mathcal{U}_{j_n}^* [(\mathbf{T})^n]_{j_n j_0} \chi_{j_0}^w u_{j_0} + \mathcal{O}_n(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \rightarrow L^2},
\end{aligned}$$

where the matrix of operators,  $\mathbf{T}$ , was defined in Lemma 5.2.

**5.3.1. Inserting nested cut-offs.** In this section we modify the Fourier integral operators  $T_{j'j}$ , taking into account that in the above expression their products are multiplied by narrow cut-offs  $\chi_j^w$ .

By construction of  $\chi_j$ , there exists  $R_0 > 0$  (independent of  $h, \tilde{h}$ ) such that for any index  $j$  the cut-off  $\chi_j \in \tilde{S}_{\frac{1}{2}}$  is supported inside the microscopic cylinder

$$(5.10) \quad B_{R_0(h/\tilde{h})^{1/2}} \stackrel{\text{def}}{=} \{(x, y, \xi, \eta) : |y|, |\eta| \leq R_0(h/\tilde{h})^{1/2}\} \subset T^*\mathbb{R}^d.$$

Fix some  $R_1 \geq 2R_0$ , and choose a function  $\tilde{\chi}^0 \in C_0^\infty(\mathbb{R}^{2d_t}, [0, 1])$  equal to unity in the ball  $\{|\tilde{y}|, |\tilde{\eta}| \leq R_1\}$ , and supported in  $\{|\tilde{y}|, |\tilde{\eta}| \leq 2R_1\}$ . Normal hyperbolicity implies that there exists  $\Lambda > 2$  such that the cylinders  $B_\bullet$  (see (5.10)) satisfy

$$(5.11) \quad \kappa_{j'j}(B_{2R}) \Subset B_{R\Lambda},$$

for all  $0 < R < 1$  and any physical pair  $j'j$ .

We then define the families of nested<sup>1</sup> cut-offs  $\{\chi^n\}_{n \in \mathbb{N}}, \{\tilde{\chi}^n\}_{n \in \mathbb{N}}$  as follows:

$$(5.12) \quad \forall n \in \mathbb{N}, \quad \tilde{\chi}^n(y, \eta) \stackrel{\text{def}}{=} \tilde{\chi}^0(y\Lambda^{-n}, \eta\Lambda^{-n}),$$

$$(5.13) \quad \chi^n(x, y, \xi, \eta) \stackrel{\text{def}}{=} \tilde{\chi}^n(y(\tilde{h}/h)^{1/2}, \eta(\tilde{h}/h)^{1/2}) \in \tilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d),$$

---

<sup>1</sup>Below we use the notation  $\chi^0 \succ \chi$  for nested cut-offs, meaning that  $\chi^0 \equiv 1$  near  $\text{supp}(\chi)$ .

uniformly in  $n$ : the smoothness of  $\chi^n$  actually improves when  $n$  grows. From the assumption  $R_1 > R_0$  we draw the nesting  $\chi^0 \succ \chi_j$  for any  $j = 1, \dots, J$ . Furthermore, the property (5.11) implies that

$$(5.14) \quad \text{for any physical pair } j'j, \quad \chi^{n+1} \succ \chi^n \circ \kappa_{j'j}.$$

From these nesting properties and from (3.7) we easily obtain the following

**Lemma 5.4.** *For any  $j = 1, \dots, J$  we get*

$$(5.15) \quad (\chi^0)^w \chi_j^w = \chi_j^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}, \quad \chi_j^w (\chi^0)^w = \chi_j^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

*In addition, uniformly for all  $n \in \mathbb{N}$ ,  $j, j' = 1, \dots, J$ ,*

$$(5.16) \quad T_{j'j}(\chi^n)^w = (\chi^{n+1})^w T_{j'j}(\chi^n)^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

We will actually only use  $n$  smaller than  $M \log 1/\tilde{h}$  for some  $M > 0$  independent of  $\tilde{h}$ ,  $h$ , so our cut-offs  $\chi^n$  will all be localized in microscopic neighbourhoods of  $\mathcal{K}$  when  $h \rightarrow 0$ . Furthermore, for such a logarithmic time the number of terms in the sum in the right hand side of (5.9) is bounded above by  $J^n \leq \tilde{h}^{-N}$  for some  $N > 0$ . As a result, taking into account the above cut-off insertions, this sum can be rewritten as

$$(5.17) \quad \Pi e^{-int_0 P/h} \chi^w u \\ = \sum_{j_n, \dots, j_0} \Pi_{j_n} \mathcal{U}_{j_n}^* T_{j_n j_{n-1}} (\chi^{n-1})^w \dots T_{j_2 j_1} (\chi^1)^w T_{j_1 j_0} (\chi^0)^w \chi_{j_0} u_{j_0} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

In the next section we will carefully analyze the kernels of the operators  $T_{j'j}(\chi^k)^w$ .

**5.4. Structure of the local phase function.** To analyze the Fourier integral operators we will examine the structure of the generating function for the symplectomorphism  $\kappa_{j_1 j_0}$ .

We start by studying the transverse linearization  $d_\perp \kappa(\rho)$  of the map  $\kappa = \kappa_{j_1 j_0}$ , for a point  $\rho \in \mathcal{K} \cap D_{j_1 j_0}$ . In our symplectic coordinate frames, this transverse map is represented by the symplectic matrix  $S_{j_1 j_0}(\rho) = S(\rho) \in \text{Sp}(2d_t, \mathbb{R})$  given by

$$(5.18) \quad S(\rho) \stackrel{\text{def}}{=} \frac{\partial(y^1, \eta^1)}{\partial(y^0, \eta^0)}(\rho), \quad \rho \in \mathcal{K}.$$

The linear symplectomorphism  $S(\rho)$  admits a quadratic generating function  $Q_\rho(y^1, y^0, \theta')$ , where  $\theta' \in \mathbb{R}^{d_t}$  is an auxiliary variable: the graph of the map  $T(y^0, \eta^0) \mapsto T(y^1, \eta^1) = S(\rho)^T(y^0, \eta^0)$  can be obtained by identifying the critical set

$$C_{Q_\rho} = \{(y^1, y^0, \theta') : \partial_{\theta'} Q(y^1, y^0, \theta') = 0\} \subset \mathbb{R}^{3d_t}.$$

This critical set is in bijection with the graph of  $S(\rho)$  through the rules

$$\eta^1 = \partial_{y^1} Q_\rho(y^1, y^0, \theta'), \quad \eta^0 = -\partial_{y^0} Q_\rho(y^1, y^0, \theta'), \quad (y^1, y^0, \theta') \in C_{Q_\rho}.$$

More structure comes from taking the normal hyperbolicity into account. Recall that our coordinates are chosen so that that  $E^+$  and  $E^-$  are  $\epsilon$ -close to  $\{\eta = 0\}$  and  $\{y = 0\}$ ,

respectively. (Here we identified  $E^\pm$  with their images under  $d\kappa_j$  – see §5.1.) This implies the existence of a symplectic transformation  $R(\rho) \in \text{Sp}(2d_t, \mathbb{R})$  such that

$$(5.19) \quad R(\rho)(\{\eta = 0\}) = E_\rho^+, \quad R(\rho)(\{y = 0\}) = E_\rho^-, \quad R(\rho) = I + \mathcal{O}(\epsilon).$$

Since  $d_\perp \kappa(\rho) \equiv S(\rho)$  maps  $E_\rho^\pm$  to  $E_{\kappa(\rho)}^\pm$ , the matrix

$$(5.20) \quad \tilde{S}(\rho) \stackrel{\text{def}}{=} R(\kappa(\rho))^{-1} S(\rho) R(\rho), \quad \rho \in \mathcal{K} \cap D_{j_1 j_0},$$

is block-diagonal:

$$(5.21) \quad \tilde{S}(\rho) = \begin{pmatrix} \Lambda(\rho) & 0 \\ 0 & {}^T \Lambda(\rho)^{-1} \end{pmatrix}.$$

The normal hyperbolicity (1.17) implies that, provided  $t_0$  has been chosen large enough<sup>2</sup>, the matrix  $\Lambda(\rho)$  is expanding, uniformly with respect to  $\rho$ :

$$(5.22) \quad \exists \nu > 0, \quad \forall \rho \in \mathcal{K}, \quad \|\Lambda^{-1}(\rho)\| \leq e^{-\nu} < 1.$$

More precisely, for any small  $\varepsilon > 0$ , if  $t_0$  is chosen large enough the coefficient  $\nu$  can be taken of the form  $\nu = t_0(\lambda_{\min} - \varepsilon)$ , where  $\lambda_{\min} > 0$  is the smallest positive transverse Lyapunov exponent of  $\varphi_t$  near  $K^\delta$ .

Combining (5.19), (5.20) and (5.21) gives

$$(5.23) \quad S(\rho) = \begin{pmatrix} \Lambda(\rho) + \mathcal{O}(\epsilon\Lambda) & \mathcal{O}(\epsilon\Lambda) \\ \mathcal{O}(\epsilon\Lambda) & \mathcal{O}(\epsilon^2\Lambda + {}^T \Lambda(\rho)^{-1}) \end{pmatrix}, \quad \rho \in \mathcal{K}.$$

This explicit form, more precisely the fact that the upper left block is invertible, allows to use a special type of quadratic generating function:

**Lemma 5.5.** *If  $t_0$  is chosen large enough, for each  $\rho$  the generating function  $Q_\rho(y^1, y^0, \theta')$  can be chosen in the following form*

$$(5.24) \quad Q_\rho(y^1, y^0, \theta') = q_\rho(y^1, \theta') - \langle y^0, \theta' \rangle.$$

*For any point  $(y^1, y^0, \theta')$  on the critical set  $C_{Q_\rho}$ , the auxiliary variable  $\theta'$  is identified with  $\eta^0$  of the corresponding phase space point.*

The function  $q_\rho(y^1, \eta^0)$  can be written in terms of a symmetric matrix  $H(\rho)$ :

$$(5.25) \quad q_\rho(y^1, \eta^0) = \frac{1}{2} \langle (y^1, \eta^0), {}^T H(\rho) (y^1, \eta^0) \rangle, \quad H(\rho) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad H_{12} \text{ invertible}.$$

The matrix  $S(\rho)$  is related to  $H(\rho)$  in the following way:

$$(5.26) \quad S(\rho) = \begin{pmatrix} H_{21}^{-1} & -H_{21}^{-1} H_{22} \\ H_{11} H_{21}^{-1} & H_{12} - H_{11} H_{21}^{-1} H_{22} \end{pmatrix}.$$

---

<sup>2</sup>Recall that  $\kappa$  represents  $\varphi_{t_0}$ .

Comparing with (5.23) we see that

$$(5.27) \quad H_{12}^T = H_{21} = \Lambda(\rho)^{-1} + \mathcal{O}(\epsilon \Lambda(\rho)^{-1}), \quad H_{11} = \mathcal{O}(\epsilon), \quad H_{22} = \mathcal{O}(\epsilon),$$

uniformly with respect to  $\rho$ . The quadratic phase function  $Q_\rho$  will be relevant when we consider the metaplectic operator  $M(\rho)$  quantizing  $S(\rho)$  in §5.5.4 (see also (3.22)).

From the study of the linearized flow in the transverse direction, we now consider the dynamics of

$$(5.28) \quad \tilde{\kappa} = \tilde{\kappa}_{j_1 j_0} : D_{j_1 j_0} \cap \mathcal{K} \longrightarrow A_{j_1 j_0} \cap \mathcal{K}.$$

along the trapped set – see Fig. 2. When no confusion is likely to arise we use the notation  $D_\bullet$  and  $A_\bullet$  for the corresponding subsets of  $\mathcal{K}$ . There we have *no* assumptions on the flow, except for it being symplectic.

Possibly after refining the covers  $U_j$ , each map  $\tilde{\kappa}$  can be generated by a nondegenerate phase function  $\psi = \psi_{j_1 j_0}(x^1, x^0, \theta)$  defined in a neighbourhood of the origin in  $\mathbb{R}^{d-d_t} \times \mathbb{R}^{d-d_t} \times \mathbb{R}^k$ , where  $0 \leq k \leq n$  — see §3.3.

Since the  $U_j$  have been chosen small, this map  $C_\psi \rightarrow \Gamma_{\tilde{\kappa}}$  can be assumed to be injective. Notice that the values of  $\psi$  away from  $C_\psi$  are irrelevant.

We now want to extend  $\psi$  into a generating function of the map  $\kappa$ , at least in a small neighbourhood of  $\mathcal{K}$ . The intuitive idea is to “glue together” the generating function  $\psi$  for  $\tilde{\kappa}$ , with the quadratic generating functions  $Q_\rho$  for the transverse dynamics  $d_\perp \kappa(\rho)$ .

Let us consider the following Ansatz for a generating function  $\Psi$  for  $\kappa$ :

$$(5.29) \quad \Psi(x^1, x^0, \theta; y^1, y^0, \theta') = \psi(x^1, x^0, \theta) + \delta\Psi(x^1, x^0, \theta; y^1, y^0, \theta'),$$

with an extra auxiliary variable  $\theta' \in \mathbb{R}^{d_t}$ . To simplify the notations we split the variables into longitudinal and transversal ones:

$$(5.30) \quad \rho_\parallel = (x^1, x^0, \theta), \quad \rho_\perp = (y^1, y^0, \theta').$$

**Lemma 5.6.** *The transversal correction,  $\delta\Psi(\rho_\parallel, \rho_\perp)$ , defined in (5.29) satisfies*

$$\delta\Psi(\rho_\parallel, \rho_\perp) = Q_{\rho_\parallel}(\rho_\perp) + \mathcal{O}((y^1, \theta')^3),$$

where  $Q_{\rho_\parallel}(\bullet)$  is a quadratic form of the same type as (5.24, 5.25), which depends smoothly on  $\rho_\parallel$ . If  $\rho_\parallel \in C_\psi$  corresponds to the point  $(\rho^1; \rho^0) \in \Gamma_{\tilde{\kappa}}$ , then  $Q_{\rho_\parallel} = Q_{\rho^0}$ .

In other words, the quadratic forms  $Q_{\rho_\parallel}$  extend the forms  $Q_\rho$  to a neighbourhood of  $C_\psi$ ,

*Proof.* Since  $\mathcal{K}$  is preserved by  $\kappa$  and carries the map  $\tilde{\kappa}$ , we may assume that for any  $\rho_\parallel$ , the function  $\delta\Psi(\rho_\parallel, \bullet)$  has no linear part in the variables  $\rho_\perp$ . At each point  $\rho_\parallel \in C_\psi$  (identified with some  $\rho^0 \in \mathcal{K}$ ), the quadratic part  $Q_{\rho_\parallel}(\rho_\perp)$  generates the linear transverse deviation from  $\tilde{\kappa}$  near the point  $\rho^0$ , namely  $d_\perp \kappa(\rho^0)$ . This means that  $Q_{\rho_\parallel} = Q_{\rho^0}$ , which has the form (5.24). This form could be attributed to the geometric fact that the graph



of  $d_\perp \kappa(\rho^0)$  admits  $(y^1, \eta^0)$  as coordinates (that is, this graph projects bijectively onto the  $(y^1, \eta^0)$ -plane).

This projection property locally extends to the graph of  $\kappa$ : in some neighbourhood of  $\mathcal{K}$ , the points of  $\Gamma_\kappa$  can be represented by the coordinates  $(\rho^0 = (x^0, \xi^0) \in \mathcal{K}; y^1, \eta^0)$ , where  $y^1, \eta^0 \in \text{neigh}(0)$ . This property shows that  $\delta\Psi$  can be written in the form

$$(5.31) \quad \delta\Psi(\rho_\parallel, \rho_\perp) = \delta\tilde{\Psi}(\rho_\parallel, y^1, \theta') - \langle y^0, \theta' \rangle.$$

As explained above, the quadratic part  $q_{\rho_\parallel}(\bullet)$  of  $\delta\tilde{\Psi}(\rho_\parallel; \bullet)$  must be equal, for  $\rho_\parallel \in C_\psi$ , to the corresponding  $q_{\rho^0}$  generating  $S(\rho^0)$ . The equations for  $C_\Psi$  show that, if we fix small values  $(y^1, \theta' = \eta^0)$ , then value  $\rho_\parallel$  such that  $(\rho_\parallel, y^1, y^0, \eta^0) \in C_\Psi$  is  $\mathcal{O}((y^1, \eta^0)^2)$ -close to  $C_\psi$ .  $\square$

**5.5. Structure of the propagators  $T_{j'j}$ .** From the above informations on the phase function  $\Psi = \Psi_{j'j}$ , we can the integral kernel of  $T = T_{j'j}$  defined in (5.7) and quantizing the map  $\kappa_{j'j}$ , as an oscillatory integral. The general theory of Fourier integral operators (see §3.3) tells us that its kernel takes the form

$$(5.32) \quad T(x^1, y^1; x^0, y^0) = \int_{\mathbb{R}^{L+d_t}} \frac{d\theta d\theta'}{(2\pi h)^{\frac{k+d_t+d}{2}}} a(\rho_\parallel, \rho_\perp) e^{\frac{i}{h}\Psi(\rho_\parallel, \rho_\perp)} + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

where we use the notation (5.30). Let us group the variables  $(x, y) = X$ ,  $(\xi, \eta) = \Xi$ ,  $(\theta, \theta') = \Theta$ . We may assume that the symbol  $a(X^1, X^0, \Theta)$  is supported in a small neighbourhood of the critical set  $C_\Psi$ . In particular, for small values of the transversal variables  $\rho_\perp$ ,  $a(\bullet, \rho_\perp)$  is supported near  $C_\psi$ . From (5.7), this Fourier integral operator is microlocally subunitary in  $V_{j'} \times V_j$ .

**5.5.1. Using the cut-off near  $\mathcal{K}$ .** We now take into account the cut-offs  $(\chi^k)^w$ , and study the truncated propagator  $T(\chi^k)^w$  appearing in (5.17).

**Lemma 5.7.** *For any  $k \geq 0$  we have*

$$(5.33) \quad T(\chi^k)^w = T\chi^k + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

where the Schwartz kernel of the operator  $T\chi^k$  is given by

$$(5.34) \quad T\chi^k(x^1, y^1; x^0, y^0) \stackrel{\text{def}}{=} \int \frac{d\theta d\eta^0}{(2\pi h)^{\frac{k+d_t+d}{2}}} a(\rho_\parallel, \rho_\perp) \chi^{\sharp(k+1)}(y^1) \chi^k(y^0, \eta^0) e^{\frac{i}{h}\Psi(\rho_\parallel, \rho_\perp)},$$

where  $\chi^{\sharp k} \stackrel{\text{def}}{=} \chi_{\eta=0}^k$ , with  $\chi^k$  given in (5.12).

*Proof.* As in (5.16), the nesting property  $\chi^{\sharp(k+1)} \succ \chi^k \circ \kappa_{j'j}$  implies that

$$(5.35) \quad (\chi^{\sharp(k+1)})^w T(\chi^k)^w = T(\chi^k)^w + \mathcal{O}(\tilde{h}^\infty),$$

uniformly for all  $k \geq 0$ . The Fourier integral operator calculus in the class  $\widetilde{S}_{\frac{1}{2}}$  presented in Lemma 3.1 has the following consequence:

$$(\chi^{\sharp(k+1)})^w T(\chi^k)^w = T\chi^k + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}),$$

which combined with (5.35) gives (5.33).  $\square$

**5.5.2. Rescaling the transversal coordinates.** Since we work at distances  $\sim (h/\tilde{h})^{\frac{1}{2}}$  from the trapped set, it will be convenient to use the rescaled transversal variables

$$(5.36) \quad \tilde{y} = (\tilde{h}/h)^{\frac{1}{2}}y, \quad \tilde{\eta} = (\tilde{h}/h)^{\frac{1}{2}}\eta.$$

Our cut-offs  $\chi^k, \tilde{\chi}^k$  defined in (5.12, 5.13) are then related by  $\tilde{\chi}^\bullet(\tilde{y}, \tilde{\eta}) = \chi^\bullet(y, \eta)$ . This change of variables induces the following unitary rescaling  $\mathcal{T} : L^2(dx dy) \rightarrow L^2(dx d\tilde{y})$ :

$$(5.37) \quad \mathcal{T}u(x, \tilde{y}) \stackrel{\text{def}}{=} (h/\tilde{h})^{\frac{d_t}{2}} u(x, (h/\tilde{h})^{\frac{1}{2}}\tilde{y}) = (h/\tilde{h})^{\frac{d_t}{2}} u(x, y).$$

We recall (see for instance [43, (4.7.16)]) that

$$\mathcal{T}a^w(x, y, hD_x, hD_y)\mathcal{T}^* = \tilde{a}^w(x, \tilde{y}, hD_x\tilde{h}D_{\tilde{y}}), \quad \tilde{a}(x, \tilde{y}, \xi, \tilde{\eta}) \stackrel{\text{def}}{=} a(x, y, \xi, \eta).$$

Through this rescaling, the operator  $T\chi^k$  is transformed into

$$\tilde{T}\chi^k \stackrel{\text{def}}{=} \mathcal{T}T\chi^k\mathcal{T}^* : L^2(dxd\tilde{y}) \longrightarrow L^2(dxd\tilde{y}),$$

with Schwartz kernel

$$(5.38) \quad \begin{aligned} \tilde{T}\chi^k(x^0, \tilde{y}^0, x^1, \tilde{y}^1) &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{d_t}} \frac{d\theta}{(2\pi h)^{\frac{k+d-d_t}{2}}} \frac{d\tilde{\eta}^0}{(2\pi\tilde{h})^{d_t}} a(\rho_{\parallel}, (h/\tilde{h})^{\frac{1}{2}}\tilde{\rho}_{\perp}) \\ &\quad \times \tilde{\chi}^{\sharp(k+1)}(\tilde{y}^1) \tilde{\chi}^k(\tilde{y}^0, \tilde{\eta}^0) e^{\frac{i}{h}\psi(\rho_{\parallel}) + \delta\Psi(\rho_{\parallel}; (h/\tilde{h})^{\frac{1}{2}}\tilde{\rho}_{\perp})} \end{aligned}$$

**5.5.3. Transversal linearization.** The factor  $\tilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\tilde{\chi}^k(\tilde{y}^0, \tilde{\eta}^0)$  appearing in the integrand (5.38) allows us to simplify the above kernel. Indeed, it implies that the variables  $\tilde{\rho}_{\perp} = (\tilde{y}^1, \tilde{y}^0, \tilde{\eta}^0)$  are integrated over a set of diameter  $\sim R_1\Lambda^k$ . One can then Taylor expand the amplitude and phase function  $\delta\Psi$  in (5.38):

$$\begin{aligned} a(\rho_{\parallel}, (h/\tilde{h})^{\frac{1}{2}}\tilde{\rho}_{\perp}) e^{\frac{i}{h}\delta\Psi(\rho_{\parallel}; (h/\tilde{h})^{\frac{1}{2}}\tilde{\rho}_{\perp})} \tilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\tilde{\chi}^k(\tilde{y}^0, \tilde{\eta}^0) &= \\ (a(\rho_{\parallel}, 0) + \mathcal{O}_{\tilde{h},k}(h^{\frac{1}{2}})_{S(T^*\mathbb{R}^d)}) e^{\frac{i}{h}Q_{\rho_{\parallel}}(\tilde{\rho}_{\perp})} \tilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\tilde{\chi}^k(\tilde{y}^0, \tilde{\eta}^0). \end{aligned}$$

Since we will restrict ourselves to values  $k \leq M \log 1/\tilde{h}|$ , uniformly bounded with respect to  $h$ , we may omit to indicate the  $k$ -dependence in the remainder. As a result, up to a small error we may keep only the quadratic part of  $\delta\Psi$ , namely consider the operator with the Schwartz kernel

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^{d_t}} \frac{d\theta}{(2\pi h)^{\frac{k+d-d_t}{2}}} \frac{d\tilde{\eta}^0}{(2\pi\tilde{h})^{d_t}} a(\rho_{\parallel}, 0) \tilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\tilde{\chi}^k(\tilde{y}^0, \tilde{\eta}^0) e^{\frac{i}{h}\psi(\rho_{\parallel})} e^{\frac{i}{h}Q_{\rho_{\parallel}}(\tilde{\rho}_{\perp})}.$$

Combining the above pointwise estimates with the fact that  $a \in S(T^*\mathbb{R}^{3d})$ , and with (5.35) and Lemma 5.7, gives

$$(5.39) \quad \begin{aligned} \tilde{T}^{\chi^k} &= \tilde{T}(\tilde{\chi}^k)^w(\tilde{y}, \tilde{h}D_{\tilde{y}}) + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}, \\ \tilde{T}(x^1, y^1; x^0, y^0) &= \int \frac{d\theta}{(2\pi h)^{\frac{k+d-d_t}{2}}} \frac{d\tilde{\eta}^0}{(2\pi \tilde{h})^{d_t}} a(\rho_{\parallel}, 0) e^{\frac{i}{\hbar}\psi(\rho_{\parallel})} e^{\frac{i}{\hbar}Q_{\rho_{\parallel}}(\tilde{\rho}^\perp)} \end{aligned}$$

uniformly for  $k \leq M \log 1/\tilde{h}$ .

**5.5.4. Factoring out the transversal contribution.** For each  $\rho_{\parallel} \in \text{supp } a(\bullet, 0)$ , the quadratic phase  $Q_{\rho_{\parallel}}(\bullet)$  generates a symplectic transformation  $S(\rho_{\parallel})$  (which, in the case  $\rho_{\parallel} \in C_\psi$  corresponds coincides with the transformation  $S(\rho^0)$  of (5.18)). As already shown in (3.22), this phase allows to represent the metaplectic operator  $M(\rho_{\parallel}) : L^2(d\tilde{y}) \rightarrow L^2(d\tilde{y})$  which  $\tilde{h}$ -quantizes this symplectomorphism:

$$(5.40) \quad M(\rho_{\parallel})(\tilde{y}^1, \tilde{y}^0) \stackrel{\text{def}}{=} (2\pi \tilde{h})^{-d_t} \int_{\mathbb{R}^{d_t}} \det(H_{12}(\rho_{\parallel}))^{1/2} e^{\frac{i}{\hbar}Q_{\rho_{\parallel}}(\tilde{\rho}^\perp)} d\tilde{\eta}^0,$$

where  $H_{12}(\rho_{\parallel})$  is the block matrix appearing in  $Q_{\rho_\ell}$ , similarly as in (5.24, 5.25).

**Remark 5.8.** In the expression (5.40) we implicitly chose a sign for the square root of  $\det(H_{12}(\rho_{\parallel}))$ . Indeed, the metaplectic representation of the symplectic group is 1-to-2, a given symplectic matrix  $S$  being quantized into two possible operators  $\pm M$ . The relations (5.27) and the uniform expansion property (5.22) show that  $\det(H_{12}(\rho_{\parallel}))$  does not vanish on the support of the amplitude  $a(\bullet, 0)$  (which is a small neighbourhood of  $C_\psi \times \{\tilde{y}^0 = \tilde{y}^1 = \tilde{\eta}^0 = 0\}$ ), so we may fix the sign in each connected component of this support. This remark will be relevant in §5.6.

Defining the symbol

$$\tilde{a}(\rho_{\parallel}) \stackrel{\text{def}}{=} \frac{a(\rho_{\parallel}, 0)}{\det(H_{12}(\rho_{\parallel}))^{\frac{1}{2}}},$$

we interpret the operator  $\tilde{T}$  in (5.39) as a Fourier integral operator with an operator valued symbol,  $M(\rho_{\parallel})$ , where  $M$  is given by (5.40). That fits exactly in the framework presented in Proposition 3.5:

$$\tilde{T}(u \otimes v)(x^1, \tilde{y}^1) = (2\pi h)^{-\frac{k+d-d_t}{2}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} \tilde{a}(\rho_{\parallel}) [M(\rho_{\parallel})v](\tilde{y}^1) e^{\frac{i}{\hbar}\psi(\rho_{\parallel})} u(x^0) dx^0 d\theta.$$

We now apply Lemma 3.4 to see that

$$(5.41) \quad \tilde{T} = \text{Op}_h^w(M)T^{\parallel} + \mathcal{O}_{\tilde{h}}(h)_{\mathcal{D}^{m_n+\ell} \rightarrow \mathcal{D}^\ell},$$

where the Schwartz kernel of  $T^{\parallel}$  is given by

$$(5.42) \quad T^{\parallel}(x^0, x^1) = (2\pi h)^{-k} \int_{\mathbb{R}^k} \tilde{a}(\rho_{\parallel}) e^{\frac{i}{\hbar}\psi(\rho_{\parallel})} d\theta,$$

while the operator valued symbol  $M(\rho^1)$  is the metaplectic operator  $\tilde{h}$ -quantizing  $S(\rho^0)$ , where  $\rho^1 = \tilde{\kappa}(\rho^0)$  and  $S(\rho^0)$  is given in (5.18). We summarize these findings in the following

**Proposition 5.9.** *Suppose that the Schwartz kernel of  $T$  is given by (5.32),  $\chi^k$ ,  $\tilde{\chi}^k$  are given in (5.12), and  $\mathcal{T}$  is the unitary rescaling defined in (5.37).*

*Then for  $k \leq K(\tilde{h})$ , where  $K(\tilde{h})$  may depend on  $\tilde{h}$  but not on  $h$ ,*

$$(5.43) \quad \mathcal{T}(T(\chi^k)^w) \mathcal{T}^* = \text{Op}_h^w(M) T^\parallel (\tilde{\chi}^k)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2} + \mathcal{O}_{\tilde{h}}(h)_{L^2 \rightarrow L^2},$$

where  $T^\parallel$  is given by (5.42) and  $M(x^1, \xi^1)$  given by (5.40) with  $\rho_\parallel \in C_\psi$  determined by  $(x^1, \xi^1) = (x^1, \partial_{x^1} \psi(\rho_\parallel))$ . Here and below we use the abbreviation  $(\tilde{\chi}^k)^{\tilde{w}} \stackrel{\text{def}}{=} (\tilde{\chi}^k)^w(\tilde{y}, \tilde{h} D_{\tilde{y}})$ .

*Proof.* Lemma 5.7, (5.39), and (5.41) give (5.43) with the remainder

$$\mathcal{O}(\tilde{h}^\infty)_{L^2(dx d\tilde{y}) \rightarrow L^2(dx d\tilde{y})} + \mathcal{O}_{\tilde{h}}(h)_{L^2(dx) \otimes \mathcal{D}^{m_n} \rightarrow L^2(dx d\tilde{y})} (\tilde{\chi}^k)^{\tilde{w}}.$$

The definition of  $\tilde{\chi}^k$  in (5.12) and (3.24) show that

$$(\tilde{\chi}^k)^{\tilde{w}} = \mathcal{O}(\Lambda^{2m_n}) : L^2(d\tilde{y}) \longrightarrow \mathcal{D}^{m_n},$$

and that gives the remainder in (5.43).  $\square$

**5.6. Back to the iterated propagator.** We can now come back to (5.9) and (5.17), re-establishing the subscripts  $j_{k+1} j_k$  on the relevant objects. We rescale all the operators by conjugating them through  $\mathcal{T}$ . Fixing the limit indices  $j_0, j_n$ , we want to study the sum of operators obtained by conjugation of terms in (5.17) by  $\mathcal{T}$ :

$$(5.44) \quad \begin{aligned} \mathcal{T}[\mathbf{T}^n]_{j_n j_0} (\chi^0)^w \mathcal{T}^* &= \mathcal{T} \left( \sum_{\mathbf{j}} \prod_{k=n-1}^0 T_{j_{k+1} j_k} (\chi^k)^w \right) \mathcal{T}^* + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2} \\ &= \sum_{\mathbf{j}} \tilde{T}_{j_n j_{n-1}} (\tilde{\chi}^{n-1})^{\tilde{w}} \cdots (\tilde{\chi}^1)^{\tilde{w}} \tilde{T}_{j_1 j_0} (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2} \end{aligned}$$

where the sum runs over all possible sequences  $\mathbf{j} = j_{n-1} \dots j_1$ . A sequence (which could be thought of geometrically as a path)  $j_n \mathbf{j} j_0$  will be relevant only if it is *physical*, meaning that there exists points  $\rho \in K^\delta$  such that  $\varphi_{kt_0}(\rho) \in U_{j_k}$  for all times  $k = 0, \dots, n$  (we say that the path  $j_n \mathbf{j} j_0$  contains the trajectory of  $\rho$ ). Any unphysical sequence leads to a term of order  $\mathcal{O}(h^\infty)$ . On the other hand, for a given point  $\rho \in K^\delta$  there are usually many sequences  $\mathbf{j}$  containing its trajectory, since the neighbourhoods  $(U_j)$ 's overlap, and so do the cut-offs  $(\pi_j)$ .

We now simplify the expression (5.44), in the following way.

**Lemma 5.10.** *In the notation of (5.9) and (5.44), and for  $n \leq M \log 1/\tilde{h}$ ,*

$$(5.45) \quad \mathcal{T}[\mathbf{T}^n]_{j_n j_0} (\chi^0)^w \mathcal{T}^* = \text{Op}_h^w(M_{j_n j_0}^n) T_{j_n j_0}^{\parallel} (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

Here  $T_{j_n j_0}^{n\parallel}$  is a Fourier integral operator on  $L^2(dx)$  quantizing the map  $\tilde{\kappa}^n : V_{j_0} \rightarrow V_{j_n}$ , defined on the departure set  $D_{j_0 j_n}^n$ . For each  $\rho \in \tilde{A}_{j_n j_0}^n = \kappa^n(D_{j_n j_0}^n)$  (the arrival set) the operator valued symbol  $M_{j_n j_0}^n(\rho)$  is a metaplectic operator quantizing the symplectic map

$$S_{j_n j_0}^n((\tilde{\kappa}^n)^{-1}(\rho)) = d_\perp \kappa^n((\tilde{\kappa}^n)^{-1}(\rho)).$$

*Proof.* If we insert the approximate factorizations (5.43) in a term  $\mathbf{j}$  of the sum in the left hand side of (5.44), this term becomes

$$(5.46) \quad \text{Op}_h^w(M_{j_n j_{n-1}}) T_{j_n j_{n-1}}^\parallel (\tilde{\chi}^{n-1})^{\tilde{w}} \cdots \text{Op}_h^w(M_{j_1 j_0}) T_{j_1 j_0}^\parallel (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

We now observe that just as we inserted the cut-offs  $\chi^k$  to obtain (5.17) from (5.9) we can remove them so that each term becomes

$$(5.47) \quad \text{Op}_h^w(M_{j_n j_{n-1}}) T_{j_n j_{n-1}}^\parallel \cdots \text{Op}_h^w(M_{j_1 j_0}) T_{j_1 j_0}^\parallel (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

We can now apply Lemmas 3.3, 3.4 and Proposition 3.5 to see that

$$(5.48) \quad \begin{aligned} & \text{Op}_h^w(M_{j_n j_{n-1}}) T_{j_n j_{n-1}}^\parallel \cdots \text{Op}_h^w(M_{j_1 j_0}) T_{j_1 j_0}^\parallel = \\ & \text{Op}_h^w(M_{j_n j_{n-1} \cdots j_0}) T_{j_n \mathbf{j} j_0}^\parallel + \mathcal{O}(\tilde{h}^{-2m_d - d_t n} h)_{L^2(dx) \otimes \mathcal{D}^{2nm_d - d_t} \rightarrow L^2}, \end{aligned}$$

where we use the shorthands

$$\begin{aligned} T_{j_n j_{n-1} \cdots j_0}^\parallel &\stackrel{\text{def}}{=} T_{j_n j_{n-1}}^\parallel T_{j_{n-1} j_{n-2}}^\parallel \cdots T_{j_1 j_0}^\parallel, \\ M_{j_n j_{n-1} \cdots j_0} &\stackrel{\text{def}}{=} (M_{j_n j_{n-1}})(M_{j_{n-1} j_{n-2}} \circ \tilde{\kappa}_{j_{n-1} j_n}) \cdots (M_{j_2 j_1} \circ \tilde{\kappa}_{j_2 \cdots j_n})(M_{j_1 j_0} \circ \tilde{\kappa}_{j_1 \cdots j_n}), \\ \tilde{\kappa}_{j_k j_{k-1} \cdots j_0} &\stackrel{\text{def}}{=} \tilde{\kappa}_{j_k j_{k-1}} \circ \tilde{\kappa}_{j_{k-1} j_{k-2}} \cdots \circ \tilde{\kappa}_{j_1 j_0}. \end{aligned}$$

These expressions only make sense for physical sequences  $j_n \mathbf{j} j_0$ . The map  $\tilde{\kappa}_{j_n \mathbf{j} j_0}$  is defined on the departure set  $D_{j_n \mathbf{j} j_0}$ .

The metaplectic operator  $M_{j_n \mathbf{j} j_0}(\rho)$  quantizes the symplectomorphism  $S_{j_n \mathbf{j} j_0}(\rho^0)$ , with  $\rho = \tilde{\kappa}^n(\rho^0) \in A_{j_n \mathbf{j} j_0}$ . This symplectomorphism represents, in the charts  $V_{j_0} \rightarrow V_{j_n}$ , the transverse linearization of the flow  $\varphi_{nt_0}$  at the point  $\kappa_{j_0}^{-1}(\rho^0)$ . As a consequence, the symplectic matrix  $S_{j_n \mathbf{j} j_0}(\rho^0)$  is identical for all sequences  $j_n \mathbf{j} j_0$  containing the trajectory of  $\rho^0$ , and we call this matrix  $S_{j_n j_0}^n(\rho^0)$ . Hence, two metaplectic operators  $M_{j_n \mathbf{j} j_0}(\rho)$ ,  $M_{j_n \mathbf{j}' j_0}(\rho)$  corresponding to two different allowed sequences can at most differ by a global sign.

For all  $\rho$  in the arrival set

$$A_{j_n j_0}^n = \bigcup_{\mathbf{j}} A_{j_n \mathbf{j} j_0} = \kappa_{j_n}(\{\rho \in U_{j_n} \cap K^\delta, \varphi_{-nt_0}(\rho) \in U_{j_0}\}),$$

we choose the sign of the metaplectic operator  $M_{j_n \mathbf{j} j_0}^n(\rho)$  quantizing  $S_{j_n j_0}^n(\rho^0)$ , such that  $M_{j_n \mathbf{j} j_0}^n(\rho)$  depends smoothly on  $\rho$  on each connected component of  $A_{j_n j_0}^n$  (there is no obstruction to this fact, due to the property mentioned in the Remark 5.8: the symplectomorphisms  $S_{j_n j_0}^n(\rho)$  also have the form (5.23)). Hence, for each physical sequence  $j_n \mathbf{j} j_0$  we

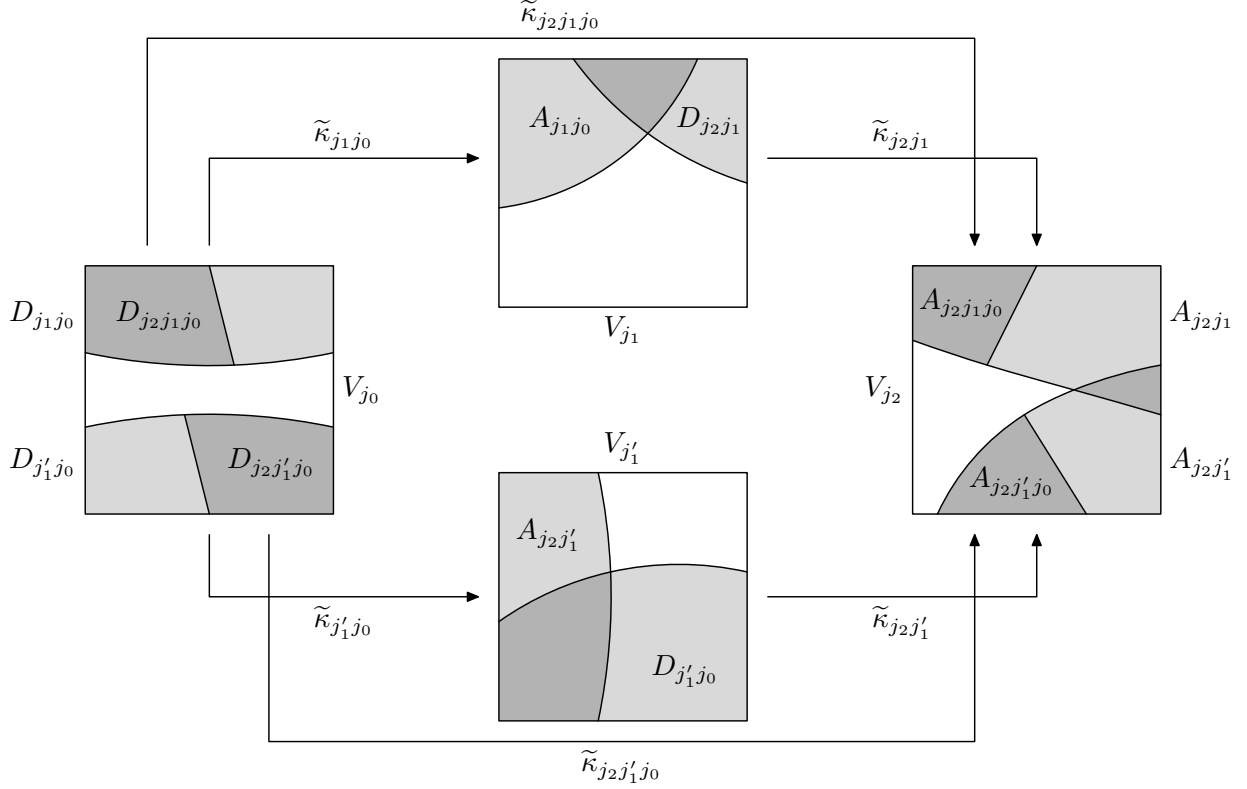


FIGURE 2. Schematic representation of the departure and arrival sets for  $\mathbf{j}$  of length 1. We show two *physical* sequences  $j_2j_1j_0$  and  $j_2j'_1j_0$  and the corresponding maps (5.28). As remarked there we use the same notation for the departure and arrival sets on  $\mathcal{K}$ .

have

$$(5.49) \quad M_{j_n \mathbf{j} j_0}(\rho) = \varepsilon_{j_n \mathbf{j} j_0}(\rho) M_{j_n j_0}^n(\rho), \quad \rho \in D_{j_n \mathbf{j} j_0},$$

for some sign  $\varepsilon_{j_n \mathbf{j} j_0}(\rho) \in \{\pm\}$  constant on each connected component of  $A_{j_n \mathbf{j} j_0}$ . As before, the functions  $\rho \mapsto \varepsilon_{j_n \mathbf{j} j_0}(\rho)$ ,  $\rho \mapsto M_{j_n \mathbf{j} j_0}(\rho)$  can be smoothly extended outside  $A_{j_n \mathbf{j} j_0}$ , into compactly supported symbols. Lemma 3.3 and the identity (5.49) give

$$(5.50) \quad \text{Op}_h^w(M_{j_n j_{n-1} \dots j_0}) T_{j_n \mathbf{j} j_0}^\parallel = \text{Op}_h^w(M_{j_n j_0}^n) (\varepsilon_{j_n \mathbf{j} j_0})^w T_{j_n \mathbf{j} j_0}^\parallel + \mathcal{O}_{\tilde{h}}(h)_{L^2(dx) \otimes \mathcal{D}^{m_{d_t}} \rightarrow L^2}.$$

When  $(\tilde{\chi}^0)^{\tilde{w}}$  is inserted in (5.48) and (5.50) we apply (3.24) to see that

$$\mathcal{O}(\tilde{h}^{-2nm_d-d_t} h)_{L^2(dx) \otimes \mathcal{D}^{2nm_d-d_t} \rightarrow L^2} (\chi^0)^{\tilde{w}} = \mathcal{O}(\tilde{h}^{-2nm_d-d_t} h)_{L^2 \rightarrow L^2} = \mathcal{O}_{\tilde{h}}(h)_{L^2 \rightarrow L^2},$$

and hence that error term can be absorbed into  $\mathcal{O}(\tilde{h}^\infty)$ .

Returning to (5.46) we see that the sum in the right hand side of (5.44) can be factorized in the following way:

$$(5.51) \quad \sum_{\mathbf{j}} \tilde{T}_{j_n j_{n-1}}(\tilde{\chi}^{n-1})^{\tilde{w}} \cdots \tilde{T}_{j_1 j_0}(\tilde{\chi}^0)^{\tilde{w}} = \text{Op}_h^w(M_{j_n j_0}^n) \left( \sum_{\mathbf{j}} T_{j_n j_{j_0}}^{\parallel} (\varepsilon_{j_n j_{j_0}})^w \right) (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2},$$

with a uniform remainder for  $n \leq M \log 1/\tilde{h}$ . Let us put  $T_{j_n j_0}^{n\parallel} \stackrel{\text{def}}{=} \sum_{\mathbf{j}} T_{j_n j_{j_0}}^{\parallel} (\varepsilon_{j_n j_{j_0}})^w$ , so that the above identity reads exactly like in the statement of the Lemma. The operator  $T_{j_n j_0}^{n\parallel}$  is sum of Fourier integral operators  $T_{j_n j_{j_0}}^{\parallel}$  defined with different phase functions  $\psi_{j_n j_{j_0}}$ , yet these phases generate (on different parts of phase space) the same map  $\tilde{\kappa}^n : D_{j_n j_0}^n \rightarrow A_{j_n j_0}^n$ . Hence,  $T_{j_n j_0}^{n\parallel}$  is an Fourier integral operator quantizing  $\tilde{\kappa}^n$ . This completes the proof of (5.45).  $\square$

The next lemma shows that the Fourier integral operator  $T_{j_n j_0}^{n\parallel}$  is essentially subunitary.

**Lemma 5.11.** *Let  $M > 0$ . For any small  $\tilde{h} > 0$ , there exists  $h_0 = h_0(\tilde{h})$  such that, for any sequence  $\mathbf{j}$  of length  $n \leq M \log 1/\tilde{h}$  and any  $h \leq h_0(\tilde{h})$ , the operator  $T_{j_n j_0}^{n\parallel}$  satisfies the following norm estimate:*

$$(5.52) \quad \|T_{j_n j_0}^{n\parallel}\|_{L^2(dx) \rightarrow L^2(dx)} \leq 1 + \mathcal{O}(\tilde{h}).$$

*Proof.* We first note that we can bound the left hand side of (5.52) by  $\tilde{h}^{-CM}$ , for some  $C$  — that follows from a trivial estimate of the terms  $T_{j_n j_{j_0}}^{\parallel}$  in (5.51).

To prove (5.52) it is clearly enough to prove the bound  $\|T_{j_n j_0}^{n\parallel}(\tilde{\chi}^0)^{\tilde{w}}\|_{L^2(dx d\tilde{y}) \rightarrow L^2(dx d\tilde{y})} \leq 1 + \mathcal{O}(\tilde{h}^\infty)$ . From Lemma 5.2 we know that  $\|\mathbf{T}^n\|_{(L^2)^J \rightarrow (L^2)^J} \leq 1 + \mathcal{O}(h)$ , which implies that  $\|[\mathbf{T}^n]_{j_0 j_n}\|_{L^2 \rightarrow L^2} \leq 1 + \mathcal{O}(h)$ . Lemma 5.10 then shows that

$$(5.53) \quad \|\text{Op}_h^w(M_{j_n j_0}^n) T_{j_n j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}}\|_{L^2 \rightarrow L^2} \leq 1 + \mathcal{O}(\tilde{h}^\infty).$$

The family of unitary metaplectic operators  $\rho \mapsto M_{j_n j_0}^n(\rho)^{-1}$  is well defined for  $\rho$  in the neighbourhood of the arrival set  $A_{j_n j_0}^n$ , and  $T_{j_n j_0}^{n\parallel}$  is microlocalized in any small neighbourhood of  $A_{j_n j_0}^n \times D_{j_n j_0}^n \subset V_{j_n} \times V_{j_0}$ . Lemma 3.3 and (3.24) then show that

$$\begin{aligned} T_{j_n j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}} &= \text{Op}_h^w((M_{j_n j_0}^n)^{-1}) \text{Op}_h^w(M_{j_n j_0}^n) T_{j_n j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}_{\tilde{h}}(h \|T_{j_n j_0}^{n\parallel}\|)_{L^2(dx) \otimes \mathcal{D}^{2m_d-d_t} \rightarrow L^2} (\tilde{\chi}^0)^w \\ &= \text{Op}_h^w((M_{j_n j_0}^n)^{-1}) \text{Op}_h^w(M_{j_n j_0}^n) T_{j_n j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}_{\tilde{h}}(h)_{L^2 \rightarrow L^2}. \end{aligned}$$

where we used the above a priori bound on  $\|T_{j_n j_0}^{n\parallel}\|$ .

Just as before we can insert the cut-off  $\tilde{\chi}^n$  (see (5.12)) with a  $\mathcal{O}(\tilde{h}^\infty)$  loss. We also introduce a cut-off  $\psi = \psi(x, \xi)$  to a small neighbourhood of  $A_{jn, j_0}$ . (It was not necessary before as  $T_{jn, j_0}^{n\parallel}$  provided the needed localization.) This and (5.53) give the bound

$$\begin{aligned} \|T_{jn, j_0}^{n\parallel}(\tilde{\chi}^0)^{\tilde{w}}\| &\leq \|\text{Op}_h^w((M_{jn, j_0}^n)^{-1}\psi)(\tilde{\chi}^n)^{\tilde{w}}\| \|\text{Op}_h^w(M_{jn, j_0}^n)T_{jn, j_0}^{n\parallel}(\tilde{\chi}^0)^{\tilde{w}}\| + \mathcal{O}(\tilde{h}^\infty) \\ &\leq \|\text{Op}_h^w((M_{jn, j_0}^n)^{-1}\psi)(\tilde{\chi}^n)^{\tilde{w}}\|(1 + \mathcal{O}(\tilde{h}^\infty)) + \mathcal{O}(\tilde{h}^\infty). \end{aligned}$$

Since by Lemma 3.3 and (3.24)

$$[\text{Op}_h^w((M_{jn, j_0}^n)^{-1}\psi)(\tilde{\chi}^n)^{\tilde{w}}]^* \text{Op}_h^w((M_{jn, j_0}^n)^{-1}\psi)(\tilde{\chi}^n)^{\tilde{w}} = [\psi^w]^* \psi^w [(\tilde{\chi}^n)^{\tilde{w}}]^* (\tilde{\chi}^n)^{\tilde{w}} + \mathcal{O}_{\tilde{h}}(h)_{L^2 \rightarrow L^2},$$

we have

$$\|\text{Op}_h^w((M_{jn, j_0}^n)^{-1}\psi)(\tilde{\chi}^n)^{\tilde{w}}\| \leq \|\psi^w\| \|(\tilde{\chi}^n)^{\tilde{w}}\| + \mathcal{O}_{\tilde{h}}(h) \leq 1 + \mathcal{O}(\tilde{h}),$$

and the bound (5.52) follows.  $\square$

**5.7. Inserting the final cut-off.** We now return to the operator  $\chi^w e^{-itn_0 P/h} \chi^w$ . From Lemma 5.3 we easily obtain

$$\begin{aligned} \chi^w e^{-itn_0 P/h} \chi^w u &= \sum_{jn, j_0} \Pi_{jn} \mathcal{U}_{jn}^* \chi_{jn}^w [(\mathbf{T})^n]_{jn, j_0} \chi_{j_0}^w u_{j_0} + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}}) \\ (5.54) \quad &= \sum_{jn, j_0} \Pi_{jn} \mathcal{U}_{jn}^* \chi_{jn}^w (\chi^0)^w [(\mathbf{T})^n]_{jn, j_0} (\chi^0)^w \chi_{j_0}^w u_{j_0} + \mathcal{O}(\tilde{h}^\infty), \end{aligned}$$

where in the first line we used (5.8), while in the second line we used (5.15). Hence our last step will consist in estimating the norm of the operator  $(\chi^0)^w [\mathbf{T}^n]_{jn, j_0} (\chi^0)^w$  (or its conjugate through  $\mathcal{T}$ ). To this aim we will use Lemma 3.4, Proposition 3.5 and the factorization (5.45) to obtain

$$\begin{aligned} (\tilde{\chi}^0)^{\tilde{w}} \mathcal{T} [\mathbf{T}^n]_{jn, j_0} \mathcal{T}^* (\tilde{\chi}^0)^{\tilde{w}} &= (\tilde{\chi}^0)^{\tilde{w}} \text{Op}_h^w(M_{jn, j_0}^n) T_{jn, j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2} \\ (5.55) \quad &= T_{jn, j_0}^{n\parallel} (\tilde{\chi}^0)^{\tilde{w}} \text{Op}_h^w(N_{jn, j_0}^n) (\tilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}. \end{aligned}$$

Here the operator valued symbol  $N_{jn, j_0}^n(\rho) = M_{jn, j_0}^n((\tilde{\kappa}^n)^{-1}(\rho))$ ,  $\rho \in D_{jn, j_0}^n$ , is a metaplectic operator quantizing the symplectic map  $S_{jn, j_0}^n(\rho) = d_\perp \kappa^n(\rho)$ . (Having it on the right now makes the notation slightly less cumbersome.)

In Lemma 5.11 we control the norm of  $T_{jn, j_0}^{n\parallel}$ . There remains to control the norm of the factor  $(\tilde{\chi}^0)^{\tilde{w}} \text{Op}_h^w(N_{jn, j_0}^n) (\tilde{\chi}^0)^{\tilde{w}}$ . For that it is enough to control the operator-valued symbol  $\text{Op}_{h, \tilde{y}}^w(\tilde{\chi}^0) N_{jn, j_0}^n(\rho) \text{Op}_{h, \tilde{y}}^w(\tilde{\chi}^0)$ .

**5.7.1. Controlling the symbol.** In (5.19) we defined, for each point  $\rho \in \mathcal{K} \cap D_{j_1, j_0}$ , a symplectic transformation  $R(\rho) \in \text{Sp}(2d_t, \mathbb{R})$  which maps the  $y$ -space to  $E_\rho^+$  and the  $\tilde{\eta}$ -space to  $E_\rho^-$ . This transformation is  $\epsilon$ -close to the identity and in particular it is uniformly bounded with respect to  $\rho$ .



By iteration of this property, for any  $\rho_0 \in D_{j_n j_0}^n$ , the map

$$\tilde{S}_{j_n j_0}^n(\rho_0) \stackrel{\text{def}}{=} R(\rho_n)^{-1} S_{j_n j_0}^n(\rho_0) R(\rho_0)$$

is block-diagonal in the basis  $(y, \eta)$ :

$$(5.56) \quad \tilde{S}_{j_n j_0}^n(\rho_0) = \begin{pmatrix} \Lambda^n(\rho_0) & 0 \\ 0 & {}^T \Lambda^n(\rho_0)^{-1} \end{pmatrix},$$

where  $\Lambda^n(\rho_0)$  is expanding. We may quantize  $R(\rho)$  into metaplectic operators  $A(\rho)$ , and define

$$\tilde{N}_{j_n j_0}^n(\rho_0) \stackrel{\text{def}}{=} A(\rho_n)^{-1} N_{j_n j_0}^n(\rho_0) A(\rho_0)$$

which quantizes  $\tilde{S}_{j_n j_0}^n(\rho_0)$ .

We can then rewrite

$$(\tilde{\chi}^0)^{\tilde{w}} N_{j_n j_0}^n(\rho) (\tilde{\chi}^0)^{\tilde{w}} = (\tilde{\chi}^0)^{\tilde{w}} A(\rho_n) \tilde{N}_{j_n j_0}^n(\rho_0) A(\rho_0)^{-1} (\tilde{\chi}^0)^{\tilde{w}}.$$

We are interested in the  $L^2 \rightarrow L^2$  norm of this operator. Since metaplectic operators are unitary, and using the covariance of the Weyl quantization with respect to metaplectic operators, this norm is equal to that of

$$(\tilde{\chi}_{\rho_n}^0)^{\tilde{w}} (\tilde{\chi}_{\rho_0}^0 \circ \tilde{S}_{j_n j_0}^n(\rho_0)^{-1})^{\tilde{w}}, \quad \tilde{\chi}_{\rho_n}^0 \stackrel{\text{def}}{=} \tilde{\chi}^0 \circ R(\rho_n), \quad \tilde{\chi}_{\rho_0}^0 \stackrel{\text{def}}{=} \tilde{\chi}^0 \circ R(\rho_0).$$

The block diagonal form of  $\tilde{S}_{j_n j_0}^n(\rho_0)$  shows that

$$(5.57) \quad [\tilde{\chi}_{\rho_0}^0 \circ (\tilde{S}_{j_n j_0}^n(\rho_0))^{-1}](\tilde{y}, \tilde{\eta}) = \tilde{\chi}_{\rho_0}^0(\Lambda^n(\rho_0)^{-1} \tilde{y}, {}^T \Lambda^n(\rho_0) \tilde{\eta}).$$

We may now invoke the following simple

**Lemma 5.12.** *Suppose that  $A$  is a  $m \times m$  real invertible matrix and that  $\chi_1, \chi_2 \in \mathcal{S}(\mathbb{R}^{2m})$ . Then*

$$(5.58) \quad \|\chi_1^w(x, \tilde{h} D_x) \chi_2^w(Ax, {}^T A^{-1} \tilde{h} D_x)\|_{L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)} \leq C |\det A|^{\frac{1}{2}} \tilde{h}^{-\frac{m}{2}},$$

where  $C$  depends on certain seminorms of  $\chi_1$  and  $\chi_2$ , but not on  $A$ .

We remark that the upper bound becomes nontrivial only if  $|\det A| \ll \tilde{h}^{m/2}$ . In that case, one cannot apply the  $\tilde{h}$ -symbol calculus any longer (which would predict that the norm is equal to  $\max_{x, \xi} |\chi_1(x, \xi) \chi_2(Ax, {}^T A^{-1} \xi)| + \mathcal{O}(\tilde{h})$ ) because the second factor is not a symbol in the class  $S(\mathbb{R}^{2m})$ .

*Proof.* If we put  $\hat{\chi}_j(x, Z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^m} \chi_j(x, \xi) e^{i\langle Z, \xi \rangle} d\xi$ , then the kernel of the operator in the lemma is given by

$$\begin{aligned} K(x, y) &= \frac{1}{(2\pi\tilde{h})^{2m}} \int_{\mathbb{R}^{3m}} \chi_1\left(\frac{x+z}{2}, \xi\right) \chi_2\left(\frac{Az+Ay}{2}, {}^T A^{-1}\eta\right) e^{i\langle x-z, \xi \rangle / \tilde{h} + i\langle z-y, \eta \rangle / \tilde{h}} d\xi d\eta dz \\ &= \frac{|\det A|}{(2\pi\tilde{h})^{2m}} \int_{\mathbb{R}^m} \hat{\chi}_1\left(\frac{x+z}{2}, \frac{x-z}{\tilde{h}}\right) \hat{\chi}_2\left(\frac{Az+Ay}{2}, \frac{Az-Ay}{\tilde{h}}\right) dz. \end{aligned}$$

We will estimate the norm using Schur's Lemma and hence we need to show that

$$(5.59) \quad \left( \max_{x \in \mathbb{R}^m} \int |K(x, y)| dy \right) \left( \max_{y \in \mathbb{R}^m} \int |K(x, y)| dx \right) \leq C^2 |\det A| \tilde{h}^{-m}.$$

Making a change of variables  $Z = (x - z)/\tilde{h}$  and  $X = (x + z)/\tilde{h}$  we obtain

$$\int |K(x, y)| dx \leq C_1 (\max_{\mathbb{R}^{2m}} |\hat{\chi}_2|) |\det A| \tilde{h}^{-m} \iint |\hat{\chi}_1(X, Z)| dZ dX \leq C |\det A| \tilde{h}^{-m}.$$

To estimate the integral in  $y$  let

$$F(Z) = \max_{\mathbb{R}^m} |\hat{\chi}_1(\bullet, Z)|, \quad G(Y) = \max_{\mathbb{R}^m} |\hat{\chi}_2(\bullet, Y)|,$$

noting that our assumptions give  $F(Z) = \mathcal{O}(\langle Z \rangle^{-\infty})$ ,  $G(Y) = \mathcal{O}(\langle Y \rangle^{-\infty})$ . Changing variables to  $Z = (x - z)/\tilde{h}$  and  $Y = (Az - Ay)/\tilde{h}$  we obtain,

$$\int |K(x, y)| dy \leq C_3 \iint F(Z) G(Y) dZ dY \leq C.$$

This proves the upper bound (5.58).  $\square$

Applying Lemma 5.12 to the product (5.57) we get the bound

$$\|(\tilde{\chi}^0)^{\tilde{w}} N_{j_n j_0}^n(\rho_0) (\tilde{\chi}^0)^{\tilde{w}}\|_{L^2(d\tilde{y}) \rightarrow L^2(d\tilde{y})} \leq C(\tilde{\chi}_{\rho_0}^0, \tilde{\chi}_{\rho_n}^0) |\det \Lambda^n(\rho_0)|^{-1/2} \tilde{h}^{-d_t/2}.$$

Since the transformations  $R(\rho)$  are uniformly bounded, the prefactor  $C(\tilde{\chi}_{\rho_0}^0, \tilde{\chi}_{\rho_n}^0)$  is uniformly bounded with respect to  $\rho_0$ . On the other hand, the determinant of  $\Lambda^n(\rho_0)^{-1}$  can be bounded as follows.

**Lemma 5.13.** *Take  $\varepsilon > 0$  arbitrary small. Then there exists  $C_\varepsilon > 0$  such that,*

$$\forall n \geq 1, \forall \rho_0 \in D_{j_n j_0}^n, \quad |\det \Lambda^n(\rho_0)^{-1}| \leq C_\varepsilon e^{-(\lambda_0 - \varepsilon)nt_0},$$

where  $\lambda_0$  was defined by (1.19).

*Proof.* This follows from writing the definition of  $\lambda_0$  using the local coordinate frames.  $\square$

We have thus obtained the following upper bound:

$$(5.60) \quad \|(\tilde{\chi}^0)^{\tilde{w}} N_{j_n j_0}^n(\rho_0) (\tilde{\chi}^0)^{\tilde{w}}\|_{L^2(d\tilde{y}) \rightarrow L^2(d\tilde{y})} \leq C_\varepsilon \tilde{h}^{-d_t/2} e^{-(\lambda_0 - \varepsilon)nt_0},$$

valid for any  $n \geq 1$  and any  $\rho_0 \in D_{j_n j_0}^n$ . In particular, the time  $n$  may arbitrarily depend on  $\tilde{h}$ .

When  $n \leq M \log 1/\tilde{h}$ , for  $M > 0$  arbitrary large but independent of  $\tilde{h}$  or  $h$ , we combine this bound with (5.17), Lemma (5.11) and Lemma 5.10 to obtain the estimate (5.2), which was the goal of this section.

## 6. MICROLOCAL WEIGHTS AND ESTIMATES AWAY FROM THE TRAPPED SET

In this section we will justify the estimates described as Step 2 of the proof in §2. That will involve a quantization of the escape function  $G$  given in Proposition 4.6 with  $\epsilon = (h/\tilde{h})^{\frac{1}{2}}$ . That means that we will use the calculus described in §3.2.

**6.1. Exponential weights.** Suppose that  $g \in \mathcal{C}_c^\infty(T^*X; \mathbb{R})$  satisfies the following estimates:

$$(6.1) \quad \frac{\exp g(\rho)}{\exp g(\rho')} \leq C(1 + (\tilde{h}/h)^{\frac{1}{2}} d(\rho, \rho'))^N, \quad \partial_\rho^\alpha g = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}), \quad |\alpha| > 0,$$

for some  $N$  and  $C$ , and for some distance function  $d(\rho, \rho')$  on  $T^*X \times T^*X$  (since  $g$  is compactly supported, the estimate is independent of the choice of  $d$ ). We note that  $G$  defined in Proposition 4.6 with  $\epsilon = (h/\tilde{h})^{\frac{1}{2}}$  satisfies these assumptions.

We first recall a variant of the Bony-Chemin theorem [5, Théorème 6.4], [43, Theorem 8.6] in the form presented in [31, Proposition 3.5, (3.21), (3.22)] (as usual  $g^w = \text{Op}_h^w(g)$ ):

**Proposition 6.1.** *Suppose that  $g \in \mathcal{C}_c^\infty(T^*X)$  satisfies (6.1). Then*

$$(6.2) \quad \exp(g^w) = b^w,$$

where the symbol  $b(x, \xi)$  satisfies the bounds

$$(6.3) \quad |\partial^\alpha b(\rho)| \leq C_\alpha e^{g(\rho)} (h/\tilde{h})^{-|\alpha|/2},$$

in any local coordinates near the support of  $g$ .

If  $\text{supp } g \Subset U$ , for an open  $U \Subset T^*X$ , then

$$(6.4) \quad \partial_x^\alpha \partial_\xi^\beta (b(x, \xi) - 1) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty}), \quad (x, \xi) \in \mathbb{C}U.$$

Also, if  $A \in \Psi^{\text{comp}}(X)$ ,  $B \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X)$  and  $C \in \Psi_{\frac{1}{2}}^{\text{comp}}(X)$  then

$$(6.5) \quad \begin{aligned} e^{g^w} A e^{-g^w} &= A + i(h\tilde{h})^{\frac{1}{2}} A_1, \quad A_1 \in \tilde{\Psi}_{\frac{1}{2}}(X), \quad \text{WF}_h(A_1) \subset \text{WF}_h(A), \\ e^{g^w} B e^{-g^w} &= B + i\tilde{h} B_1, \quad B_1 \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X), \quad \text{WF}_h(B_1) \subset \text{WF}_h(B), \\ e^{g^w} C e^{-g^w} &= C + i\tilde{h}^{\frac{1}{2}} C_1, \quad C_1 \in \Psi_{\frac{1}{2}}^{\text{comp}}(X), \quad \text{WF}_h(C_1) \subset \text{WF}_h(C). \end{aligned}$$

The assumptions in (6.1) show that  $\exp g$  is an order function for the  $\tilde{S}_{\frac{1}{2}}$  calculus – see [31, §3.3, (3.17), (3.18)]. Hence we can apply composition formulae. In particular if  $g_j$ ,  $j = 1, 2$  satisfy (6.1) then

$$(6.6) \quad \exp(g_1^w) \exp(g_2^w) = c^w, \quad |\partial^\alpha c(\rho)| \leq C_\alpha \exp(g_1 + g_2) (h/\tilde{h})^{-|\alpha|/2}.$$

Because of the compact supports of  $g_j$ 's and because of (6.3) derivatives can be taken in any local coordinates.

The consequence of (6.6) useful to us here is given in the following Lemma.

**Lemma 6.2.** *Suppose that  $A \in \widetilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X)$  and that*

$$\widetilde{\sigma}(A) = a + \mathcal{O}((h\tilde{h})^{\frac{1}{2}})_{\widetilde{S}_{\frac{1}{2}}}, \quad a \in \mathcal{C}_c^\infty(T^*X) \cap \widetilde{S}_{\frac{1}{2}}(T^*X).$$

*If  $U_{h,\tilde{h}} \stackrel{\text{def}}{=} \{\rho \in T^*X : d(\rho, \text{supp } a) < (h/\tilde{h})^{\frac{1}{2}}\}$ , then*

$$(6.7) \quad \|A e^{g_1^w} e^{g_2^w}\|_{L^2 \rightarrow L^2} = \sup_{T^*X} (|a| e^{g_1+g_2}) + \mathcal{O}(\tilde{h} \sup_{U_{h,\tilde{h}}} e^{g_1+g_2}) + \mathcal{O}(h^{\frac{1}{2}} \log(1/h)).$$

*Proof.* We first consider this statement in  $\mathbb{R}^n$ . We apply the standard rescaling (3.4) noting that (6.1) imply that  $\tilde{m}_j = \tilde{g}_j$  are order functions. If  $d$  is the Euclidean distance and if we put

$$n_N(\tilde{\rho}) \stackrel{\text{def}}{=} (1 + d(\tilde{\rho}, \tilde{U}))^{-N}, \quad \tilde{U} \stackrel{\text{def}}{=} (\tilde{h}/h)^{\frac{1}{2}} U_{h,\tilde{h}},$$

then  $n_N$  is an order function for any  $N$ , and  $\tilde{a} \in S(n_N)$  for all  $N$ . We have

$$A = \text{Op}_h^w(a + (h/\tilde{h})^{\frac{1}{2}} a_1), \quad \text{for some } a_1 \in \widetilde{S}_{\frac{1}{2}},$$

and hence, after rescaling,

$$\tilde{A} e^{\text{Op}_h^w(\tilde{g}_1)} e^{\text{Op}_h^w(\tilde{g}_2)} = \text{Op}_h^w(\tilde{b}) + (h/\tilde{h})^{\frac{1}{2}} \text{Op}_h^w(\tilde{b}_1),$$

$$\tilde{b} \in S(n_N \tilde{m}_1 \tilde{m}_2), \quad \tilde{b} - \tilde{a} e^{\tilde{g}_1 + \tilde{g}_2} \in \tilde{h} S(n_N \tilde{m}_1 \tilde{m}_2), \quad \tilde{b}_1 \in S(\tilde{m}_1 \tilde{m}_2).$$

Put

$$\begin{aligned} M = M(h, \tilde{h}) &\stackrel{\text{def}}{=} \sup_{\mathbb{R}^{2n}} n_N \tilde{m}_1 \tilde{m}_2 \leq \sup_{\tilde{\rho}} \left( (1 + d(\tilde{\rho}, \tilde{U}))^{-N} e^{\tilde{g}_1(\tilde{\rho}) + \tilde{g}_2(\tilde{\rho})} \right) \\ &\leq \left( \sup_{\tilde{U}} e^{\tilde{g}_1 + \tilde{g}_2} \right) \left( 1 + \sup_{\tilde{\rho}} (1 + C_1 C_2 d(\tilde{\rho}, \tilde{U}))^{-N+N_1+N_2} \right) \\ &\leq C \sup_{U_{h,\tilde{h}}} e^{g_1+g_2}, \end{aligned}$$

where we took  $N \geq N_1 + N_2$ , with  $N_j, C_j$  appearing in (6.1) for  $g_j$ .

We now apply [43, Theorem 13.13] (with  $h$  replaced by  $\tilde{h}$ ) to  $\tilde{b}/M \in S$ . That gives

$$\|\text{Op}_h^w(\tilde{b})\| = \sup |a| e^{g_1+g_2} + \mathcal{O}(\tilde{h}) \sup_{U_{h,\tilde{h}}} e^{g_1+g_2}.$$

Since  $\tilde{m}_1 \tilde{m}_2 = \mathcal{O}(\log(1/h))$ , applying the same argument to  $\tilde{b}_1/\log(1/h)$  gives (6.7).

The calculus is invariant modulo  $\mathcal{O}((h\tilde{h})^{\frac{1}{2}})$  terms (see (3.5) and [11, §5.1], [42, §3.2]), so these local estimates on  $\mathbb{R}^n$  imply similar estimates on manifolds.  $\square$

The next result is a version of (3.6) for exponentiated weights  $g$ . It is a special case of [31, Proposition 3.14] which follows from globalization of the local result [31, Proposition 3.11]. We state it using concepts recalled in §3.3.

**Proposition 6.3.** *Suppose that  $T \in I^{\text{comp}}(X \times X, \Gamma'_\kappa)$  where  $\kappa : U_1 \rightarrow U_2$ ,  $U_j \subset T^*X$ , is a symplectomorphism, that  $g \in \mathcal{C}_c^\infty(T^*X)$  satisfies (6.1), and that  $A \in \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}$ . Then*

$$(6.8) \quad \begin{aligned} e^{g^w} AT &= Te^{(\kappa^*g)^w} B + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} T_1 e^{(\kappa^*g)^w} C, \\ T_1 &\in I_h^{\text{comp}}(X \times X, \Gamma'_\kappa), \quad B, C \in \tilde{\Psi}_{\frac{1}{2}}(X), \quad \sigma(B) = \kappa^* \sigma(A). \end{aligned}$$

**6.2. Estimates away from the trapped set.** We now provide precise versions of the estimates (2.5) and (2.6) described in the Step 2 of the proof in §2.

For the escape function  $G$  constructed in Proposition 4.6 we define the operator

$$(6.9) \quad G^w \stackrel{\text{def}}{=} \text{Op}_h^w(G) \in \log(\tilde{h}/h) \tilde{\Psi}_{\frac{1}{2}}^{\text{comp}}(X), \quad \tilde{\sigma}(G) = G + \mathcal{O}((h\tilde{h})^{\frac{1}{2}-})_{\tilde{S}_{\frac{1}{2}}}.$$

Since  $G$  satisfies (6.1), Proposition 6.1 describes the exponentiated operator  $e^{G^w} = e^{\text{Op}_h^w(G)}$ .

The first lemma shows that the weights are bounded near the trapped set:

**Lemma 6.4.** *Suppose that  $\chi \in \mathcal{C}_c^\infty(T^*X) \cap \tilde{S}_{\frac{1}{2}}(T^*X)$  has the property*

$$(6.10) \quad \text{supp } \chi \subset \{\rho \in T^*X : d(\rho, K^{2\delta}) < C_0(h/\tilde{h})^{\frac{1}{2}}\},$$

for some constant  $C_0 > 0$ .

Then for some constants  $h_0, \tilde{h}_0, C_1 > 0$  we have for  $0 < h < h_0$ ,  $0 < \tilde{h} < \tilde{h}_0$ ,

$$(6.11) \quad \|\chi^w e^{G^w}\| \leq C_1, \quad \|e^{G^w} \chi^w\| \leq C_1.$$

*Proof.* Since  $\tilde{\sigma}(\chi^w) = \chi + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{\tilde{S}_{\frac{1}{2}}}$ , and  $|G(\rho)| \leq C_2$  for  $d(\rho, K^{2\delta}) < (C_0 + 1)(h/\tilde{h})^{\frac{1}{2}}$ , the estimates in (6.11) follow directly from Lemma 6.2.  $\square$

The main result of this section provides bounds for the conjugated propagator. It relies heavily on the material about the propagator for the complex absorbing potential (CAP) modified Hamiltonian,  $\exp(-it(P - iW)/h)$ , presented in the Appendix.

**Proposition 6.5.** *Suppose that  $G^w$  is given by (6.9) and that  $A \in \Psi^{\text{comp}}(X)$  satisfy*

$$(6.12) \quad \text{WF}_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0)),$$

for some  $\epsilon_0 > 0$ .

Then for some constants  $h_0, \tilde{h}_0, C_1 > 0$  we have for  $0 < h < h_0$ ,  $0 < \tilde{h} < \tilde{h}_0$ ,

$$(6.13) \quad \|e^{-G^w} e^{-it_0(P - iW)/h} e^{G^w} A\| \leq e^{2C_1}.$$

If  $\chi$  satisfies (6.10) and in addition

$$(6.14) \quad \chi(\rho) \equiv 1 \quad \text{for } d(\rho, K^{2\delta}) < \frac{1}{2}C_0(h/\tilde{h})^{\frac{1}{2}}, \quad |p(\rho)| \leq \delta,$$

where  $C_0$  is a large constant depending on  $t_0$ , then, if  $\|A\| \leq 1$ ,

$$(6.15) \quad \|(1 - \chi^w) e^{-G^w} e^{-it_0(P - iW)/h} e^{G^w} A\| < e^{-\Gamma},$$

where  $\Gamma$  is the constant appearing in the definition (4.21) of  $G$

*Proof.* Let  $A_{-G} \stackrel{\text{def}}{=} e^{G^w} A e^{-G^w}$ . Then (6.5) in Proposition 6.1 shows that

$$(6.16) \quad A_{-G} = A + \mathcal{O}_{L^2 \rightarrow L^2}(h^{\frac{1}{2}}) = \mathcal{O}(1)_{L^2 \rightarrow L^2} \quad \text{and} \quad A_{-G} = \tilde{A} A_{-G} + \mathcal{O}(h^\infty),$$

where  $\tilde{A}$  satisfies (A.10). To prove (6.13) we use the notation of Proposition A.3, and rewrite the operator on the right hand side as

$$(6.17) \quad \begin{aligned} e^{-G^w} e^{-it_0(P-iW)/h} e^{G^w} A &= e^{-G^w} e^{-it_0 P/h} e^{G^w} e^{-G^w} V_{\tilde{A}}(t_0) A_{-G} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2} \\ &= e^{-G^w} e^{-it_0 P/h} e^{G^w} C(t_0) + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2}, \end{aligned}$$

where using (6.5) and Proposition A.3,

$$C(t_0) \in \Psi_{\frac{1}{2}}^{\text{comp}}(X), \quad \text{WF}_h(C(t_0)) \subset \text{WF}_h(A) \cap w^{-1}(0).$$

Since

$$e^{\pm G^w} = B e^{\pm G^w} + (I - B) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}, \quad \text{for some } B \in \Psi^{\text{comp}}(X),$$

Proposition 6.3 (applied with  $A \equiv I$ ) and (3.8) show that for some  $B_0 \in \tilde{\Psi}_{\frac{1}{2}}(X)$ ,

$$e^{-G^w} e^{-it_0 P/h} e^{G^w} = e^{-it_0 P/h} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} \left( I + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} B_0 \right) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

From this and (6.17) we see that to prove (6.13) it is enough to show that

$$(6.18) \quad \begin{aligned} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} B_1 &= \mathcal{O}(1)_{L^2 \rightarrow L^2}, \quad B_1 \in \Psi^{\text{comp}}(X), \\ \text{WF}_h(B_1) &\subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0)). \end{aligned}$$

Lemma 6.2 applied with  $g_1 = -\varphi_{t_0}^* G$  and  $g_2 = G$ , and the property  $G - \varphi_{t_0}^* G \leq C_7$  in (4.22) which holds in a neighbourhood of  $\text{WF}_h(B_1)$ , give (6.18) and hence (6.13).

To obtain (6.15) we proceed similarly but applying the property  $\varphi_{t_0}^* G - G \geq 2\Gamma$  which is valid outside a  $(h/\tilde{h})^{\frac{1}{2}}$  neighbourhood of  $K^\delta$  – see (4.22). In more detail, Proposition 6.3 applied with  $A = 1 - \chi^w$  gives

$$\begin{aligned} (1 - \chi^w) e^{-G^w} e^{-it_0(P-iW)/h} e^{G^w} A &= \\ (1 - \chi^w) e^{-G^w} e^{-it_0 P/h} e^{G^w} e^{-G^w} V_{\tilde{A}}(t_0) A_{-G} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2} &= \\ e^{-it_0 P/h} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - (\varphi_{t_0}^* \chi)^w) e^{-G^w} V_{\tilde{A}}(t_0) A_{-G} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2}, \end{aligned}$$

where we used the boundedness established in (6.13) to control the lower order terms.

Defining  $\chi_1 \stackrel{\text{def}}{=} \varphi_{t_0}^* \chi$ , we have, by the invariance of  $K^\delta$  under the flow,

$$\chi_1 \equiv 1 \quad \text{for } d(\rho, K^\delta) \leq C_1(h/\tilde{h})^{\frac{1}{2}}, \quad |p(\rho)| \leq \delta.$$

Let  $\psi \in \mathcal{C}_c^\infty(T^*X)$  be equal to 1 in the set  $W_1$  of Proposition 4.6, and  $\text{supp } \psi \subset (w^{-1}(0))^\circ$ .

Since (6.5) and Proposition A.3 give

$$\begin{aligned} \|e^{-G^w} V_{\tilde{A}}(t_0) A_{-G} e^{G^w}\| &\leq \|e^{-G^w} V_{\tilde{A}}(t_0) e^{G^w}\| \|A\| \leq \|A\| \left( \|\tilde{A}\| + \mathcal{O}_{L^2 \rightarrow L^2}(\tilde{h}^{\frac{1}{2}}) \right) \\ &\leq 1 + \mathcal{O}(\tilde{h}^{\frac{1}{2}}), \end{aligned}$$

it is enough to show that

$$(6.19) \quad \|e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - \chi_1^w) \psi^w\| \leq e^{-3\Gamma/2},$$

$$(6.20) \quad \|e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - \chi_1^w) (1 - \psi^w) B\| \leq Ch,$$

for  $B \in \Psi^{\text{comp}}(X)$  with  $\text{WF}_h(B) \subset W_2$  where  $W_2 \subset w^{-1}([0, \epsilon_0/2]) \cap p^{-1}([-\delta, \delta])$ , is as in Proposition 4.6. Both inequalities follow from Lemma 6.2 and properties of  $G$  in (4.22). For (6.19) we apply (6.7). For (6.20) we note that on

$$\varphi_{t_0}^* G - G \geq C_8 \log(\tilde{h}/h), \quad \text{on } \text{supp}(1 - \psi) \cap W_2,$$

and (6.7) gives the estimate.  $\square$

## 7. PROOF OF THEOREM 2

We first prove (2.2) which we rewrite as follows

$$(7.1) \quad \|U_G^n A\|_{L^2(X) \rightarrow L^2(X)} \leq C e^{-nt_0(\lambda_0 - \epsilon)/2}, \quad M_\epsilon \log \frac{1}{\tilde{h}} \leq n \leq M \log \frac{1}{\tilde{h}}$$

where

$$U_G \stackrel{\text{def}}{=} \exp(-it_0 \tilde{P}_G/h) A = e^{-G^w} e^{-it_0(P-iW)/h} e^{G^w},$$

with  $t_0$  chosen in previous sections, and

$$(7.2) \quad A \in \Psi^{\text{comp}}(X), \quad \text{WF}_h(A) \subset p^{-1}((-\delta, \delta)).$$

To apply the estimates of the last two sections we first observe that Proposition A.2 implies that for any  $r$  there exist  $B_j \in \Psi^{\text{comp}}$ ,  $j = 1, \dots, r$ , each satisfying (7.2), such that

$$(7.3) \quad U_G^r A = \prod_{j=1}^r U_G B_j + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}, \quad B_r = A,$$

where the constants in the norm estimate  $\mathcal{O}(h^\infty)$  depend on  $r$ . This means that, for  $r$  independent of  $h$  but depending on  $\tilde{h}$ ,  $U_G^r A$  can be replaced by the product of operators  $U_G B_j$ , to which estimates of the previous section are applicable.

We now want to decompose  $U_G^n$  in such a way that the estimates obtained in §§5,6 can be used. For that we define

$$(7.4) \quad U_G = U_{G,+} + U_{G,-}, \quad U_{G,+} \stackrel{\text{def}}{=} U_G \chi^w, \quad U_{G,-} \stackrel{\text{def}}{=} U_G (1 - \chi)^w.$$

We note that Proposition 6.1 shows that

$$\begin{aligned}
(7.5) \quad \chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} &= e^{-G^w} e^{G^w} \chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} \\
&= e^{-G^w} \chi^w e^{-it(P-W)/h} e^{G^w} + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \rightarrow L^2} \\
&= e^{-G^w} \chi^w e^{-it(P-W)/h} e^{itP/h} e^{-itP/h} e^{G^w} + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \rightarrow L^2} \\
&= e^{-G^w} \chi_t^w e^{-itP/h} e^{G^w} + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \rightarrow L^2},
\end{aligned}$$

where  $\chi_t^w \stackrel{\text{def}}{=} \chi^w e^{-it(P-W)/h} e^{itP/h}$ . We now use Proposition A.3 applied with  $P$  replaced by  $-P$ ,  $A \in \Psi^{\text{comp}}$  satisfying  $\text{WF}_h(I - A) \cap \text{WF}_h(\chi^w) = \emptyset$ . In the notation of (A.12),  $\chi_t^w = \chi^w V_A(t)^*$ ,  $V_A(t)^* \in \Psi_\delta^{\text{comp}}(X)$ . From (A.12)

$$\sigma(V_A(t)) = \exp\left(-\frac{1}{h} \int_0^t \varphi_{-s}^* W\right) \sigma(A),$$

with a full expansion of the symbol in any coordinate chart given in Lemma A.4. For  $\rho \in \text{supp } \chi$ ,  $d(\rho, K^\delta) = \mathcal{O}(h^{\frac{1}{2}})$ , and as  $K^\delta$  is invariant under the flow  $d(\varphi_{-s}(\rho), K^\delta) = \mathcal{O}_s(h^{\frac{1}{2}})$ . But that means that on the support  $\chi$ ,  $\varphi_{-s}^* W \equiv 0$  for  $s \leq t$ , where  $t$  is independent of  $h$ , as long as  $h$  is small enough. This means that  $\text{WF}_h(I - V_A(t)^*) \cap \text{WF}_h(\chi) = \emptyset$  and hence,  $\chi_t = \chi + \mathcal{O}(h^{\frac{1}{2}})_{S_{\frac{1}{2}}}$ .

Returning to (7.5) this means that for  $t \leq C \log(1/\tilde{h})$  (in fact for any time bounded independently of  $h$ ), we have

$$\begin{aligned}
(7.6) \quad \chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} &= \chi^w e^{-G^w} e^{-itP/h} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2}, \\
e^{-G^w} e^{-it(P-iW)/h} e^{G^w} \chi^w &= e^{-G^w} e^{-itP/h} e^{G^w} \chi^w + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2}.
\end{aligned}$$

Using the notation (7.4)

$$\begin{aligned}
(7.7) \quad U_G^n &= \sum_{\varepsilon_i = \pm} U_{G, \varepsilon_n} \cdots U_{G, \varepsilon_2} U_{G, \varepsilon_1} \\
&= \sum_{\epsilon \in \Sigma(n)} U_\epsilon, \quad U_\epsilon \stackrel{\text{def}}{=} U_{G, \varepsilon_n} \cdots U_{G, \varepsilon_2} U_{G, \varepsilon_1},
\end{aligned}$$

where we used the symbolic words  $\epsilon = \epsilon_1 \cdots \epsilon_t \in \Sigma(n) = (\pm)^n$ . Now, for each word  $\epsilon \neq - \cdots -$ , call  $n_L(\epsilon)$  ( $n_R(\epsilon)$ , respectively), the number of consecutive  $(-)$  starting from the left (the right, respectively):

$$\epsilon = \underbrace{- \cdots -}_{n_L(\epsilon)} + * * \cdots * * + \underbrace{- \cdots -}_{n_R(\epsilon)}.$$



Given integers  $n_L, n_R$ , call  $\Sigma(n, n_L, n_R)$  the set of words  $\epsilon \in \Sigma(n)$  such that  $n_L(\epsilon) = n_L$  and  $n_R(\epsilon) = n_R$ . The decomposition (7.7) can be split into

$$U_G^n = U_{G,-}^n + \sum_{n_L, n_R} \sum_{\epsilon \in \Sigma(n, n_L, n_R)} U_\epsilon.$$

where the sum runs over  $n_L, n_R \geq 0$  such that  $1 \leq n_L + n_R \leq n - 1$ .

We make the following observations:

$$\Sigma(n, n_L, n_R) = \{(-)^{n_L} + (-)^{n_R}\}, \quad \text{if } n_L + n_R = n - 1,$$

$$\Sigma(n, n_L, n_R) = \{(-)^{n_L} + \epsilon' + (-)^{n_R} : \epsilon' \in \Sigma(n - n_L - n_R - 2)\}, \quad \text{if } n_L + n_R < n - 1.$$

Hence, the above sum can be recast into

$$(7.8) \quad \begin{aligned} U_G^n &= U_{G,-}^n + \sum_{n_L=0}^{n-1} (U_{G,-})^{n_L} U_{G,+} (U_{G,-})^{n-n_L-1} \\ &\quad + \sum_{n_L, n_R} (U_{G,-})^{n_L} U_{G,+} (U_G)^{n-n_R-n_L-2} U_{G,+} (U_{G,-})^{n_R} \end{aligned}$$

where the last sum runs over  $n_L, n_R \geq 0$  such that  $1 \leq n_L + n_R \leq n - 2$ .

The following lemma provides the estimate for terms in the first sum on the right hand side of (7.8):

**Lemma 7.1.** *For  $\tilde{h} > h > 0$  small enough, the following bound holds for  $r_0 \leq r \leq C_0 \log(1/\tilde{h})$ ,  $r \in \mathbb{N}$ ,*

$$(7.9) \quad \|U_{G,+} U_G^r U_{G,+}\|_{L^2 \rightarrow L^2} \leq C \tilde{h}^{-d_t/2} \exp\left(-\frac{1}{2} t_0 r (\lambda_0 - \epsilon)\right),$$

where the constant  $C$  is uniform with respect to  $h$ ,  $\tilde{h}$  and  $r$ .

*Proof.* Lemma 6.4 shows that  $e^{G^w} \chi^w$ ,  $\chi^w e^{-G^w} = \mathcal{O}(1)_{L^2 \rightarrow L^2}$ . Also, Lemma 6.2 shows that for  $\chi_1 \in \tilde{S}_{\frac{1}{2}}$  with the same properties as  $\chi$  but equal to 1 on the support of  $\chi$ , we have

$$\chi^w e^{-G^w} = \chi^w e^{-G^w} \chi_1^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}, \quad e^{G^w} \chi^w = \chi_1^w e^{G^w} \chi^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \rightarrow L^2}.$$

Using (7.6) the operator on the left hand side of (7.9) can be rewritten as

$$\begin{aligned} U_G \chi^w (U_G)^{r+1} \chi^w &= U_G \chi^w e^{-G^w} e^{-i(r+1)t_0 P/h} e^{G^w} \chi^w + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2} \\ &= U_G (\chi^w e^{-G^w} \chi_1^w + \mathcal{O}(\tilde{h}^\infty)) e^{-i(r+1)t_0 P/h} (\chi_1^w e^{G^w} \chi^w + \mathcal{O}(\tilde{h}^\infty)) \\ &\quad + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \rightarrow L^2} \end{aligned}$$

Hence,

$$\begin{aligned} \|U_{G,+} U_G^r U_{G,+}\|_{L^2 \rightarrow L^2} &\leq \|U_G\| \|\chi^w e^{-G^w}\| \|\chi_1^w e^{-i(r+1)t_0 P/h} \chi_1^w\| \|e^{G^w} \chi^w\| + \mathcal{O}(\tilde{h}^\infty) \\ &\leq C \|\chi_1^w e^{-i(r+1)t_0 P/h} \chi_1^w\|, \end{aligned}$$

where we used the fact that the operators  $U_G$ ,  $e^{G^w} \chi^w$  and  $\chi^w e^{-G^w}$  are uniformly bounded on  $L^2$ . We can now apply Proposition 5.1, replacing  $t$  by  $(r+1)t_0$  and  $\chi$  by  $\chi_1$ .  $\square$

Let us now take  $n = C_0 \log 1/\tilde{h}$ , with  $C_0 \gg 1$ . We recall that  $\Gamma$  in (2.5) was assumed to satisfy  $\Gamma > t_0 \lambda_0/2$ . We will use the bounds (7.9), and Proposition 6.5:  $\|U_{G,-} A\| < e^{-\Gamma}$ .

Returning to the estimate for  $U_G^n A$  we first observe that (7.3) and the estimates (6.15) in Proposition 6.5 give

$$(7.10) \quad \|U_{G,-}^m A\| \leq e^{-m\Gamma} + \mathcal{O}_r(h^{\frac{1}{2}}).$$

In (7.8), for each  $\ell = 1, \dots, n-2$ , we group together terms with  $n_L + n_R = \ell$ , and apply Lemma 7.1 and (7.10):

$$\begin{aligned} \|U_G^n A\| &\lesssim e^{-n\Gamma} + n e^{-(n-1)\Gamma} + \tilde{h}^{-d_t/2} \sum_{\ell=1}^{n-2} \ell e^{-\ell\Gamma} e^{-t_0(n-\ell)\frac{\lambda_0-\epsilon}{2}} + \mathcal{O}(h^{\frac{1}{2}}) \\ &\lesssim n e^{-n\Gamma} + \tilde{h}^{-d_t/2} e^{-t_0 n \frac{\lambda_0-\epsilon}{2}} \sum_{\ell=1}^{n-2} \ell e^{-\ell(\Gamma - t_0 \frac{\lambda_0-\epsilon}{2})} + \mathcal{O}(h^{\frac{1}{2}}) \\ &\lesssim \tilde{h}^{-d_t/2} e^{-t_0 n \frac{\lambda_0-\epsilon}{2}}. \end{aligned}$$

By taking  $C_0 = M_\epsilon \gg 1/\epsilon$  we may absorb the prefactor  $\tilde{h}^{-d_t/2}$  and obtain, for  $\tilde{h} > 0$  small enough,

$$(7.11) \quad \|U_G^n A\| \leq C \exp\left(-nt_0 \frac{\lambda_0 - 2\epsilon}{2}\right), \quad n \approx M_\epsilon \log 1/\tilde{h}.$$

We can now complete the proof of (1.18) following the outline in §2. We first note that (7.11) gives (2.2), so that (see (2.3)) for

$$(7.12) \quad z \in [-\delta/2, \delta/2] - ih[0, (\lambda_0 - 3\epsilon)/2]$$

and  $A \in \Psi^{\text{comp}}(X)$  satisfying (7.2),

$$\begin{aligned} (\tilde{P}_G - z)Q_A(z) &= A - R(z), \quad R(z) = \mathcal{O}(\tilde{h})_{L^2 \rightarrow L^2}, \\ Q_A(z) &\stackrel{\text{def}}{=} \frac{i}{h} \int_0^{T(\tilde{h})} e^{-it(\tilde{P}_G - z)/h} A dt = \mathcal{O}\left(\frac{T(\tilde{h})}{h}\right)_{L^2 \rightarrow L^2}, \quad T(\tilde{h}) = M_\epsilon \log 1/\tilde{h}. \end{aligned}$$

We now apply this estimate with  $A \in \Psi^{\text{comp}}(X)$  such that  $\sigma(A) \equiv 1$  in  $p^{-1}(-3\delta/4, 3\delta/4) \cap w^{-1}([0, \epsilon_0])$ . Then  $\tilde{P}_G - z \in \tilde{\Psi}_{\frac{1}{2}}^m(X)$  is elliptic outside of  $\text{WF}_h(A)$ . Hence, using the  $\tilde{\Psi}_{\frac{1}{2}}$  calculus of §3.2, there exists  $\tilde{Q}_A(z) \in \tilde{\Psi}_{\frac{1}{2}}^{-m}(X)$  such that

$$(\tilde{P}_G - z)\tilde{Q}_A(z) = I - A + \tilde{R}(z), \quad \tilde{R}(z) = \mathcal{O}(\tilde{h})_{L^2 \rightarrow L^2}.$$

The Fredholm operator  $\tilde{P}_G - z$  has index 0 since  $\tilde{P}_G + i$  is invertible for small  $\tilde{h}$ . It follows that for  $\tilde{h}$  small enough and  $z$  satisfying (7.12)

$$(\tilde{P}_G - z)^{-1} = (Q_A(z) + \tilde{Q}_A(x))(I + R(z) + \tilde{R}_A(z))^{-1} = \mathcal{O}(1/h)$$

Since  $e^{\pm G^w(x, hD)} = \mathcal{O}(h^{-M/2+1})_{L^2 \rightarrow L^2}$  for some  $M$ , it follows that

$$\begin{aligned} (P - iW - z)^{-1} &= \mathcal{O}(h^{-M})_{L^2 \rightarrow L^2} \quad z \in [-\delta/2, \delta/2] - ih[0, (\lambda_0 - 3\epsilon)/2], \\ (P - iW - z)^{-1} &= \mathcal{O}(1/\operatorname{Im} z)_{L^2 \rightarrow L^2}, \quad \operatorname{Im} z > 0, \end{aligned}$$

where the second is immediate from non-negativity of  $W$  as an operator.

We now use a semiclassical maximum principle [7, Lemma 4.7], [36, Lemma 2] to obtain the bound for  $(P - iW - z)^{-1}$  in (1.18) (after adjusting  $\delta$  and  $\epsilon$ ).

**Remark 7.2.** Strictly speaking we proved (1.18) for  $z \in [-\delta/2, \delta/2] - ih[0, \lambda_0/2 - \epsilon_0]$ , for any  $\epsilon_0$ , provided that  $h$  is small enough.

## 8. THE CAP REDUCTION OF SCATTERING PROBLEMS: PROOF OF THEOREM 3

In this section we will prove a generalization of Theorem 3 which applies to a variety of scattering problems. Our approach of reduction to estimates for the Hamiltonian complex absorbing potential (CAP) is based on the work Datchev-Vasy [13] (see also [20, §4.1] and [15]) but as the argument is simple and elegant we reproduce it in our slightly modified setting.

Let  $(Y, g)$  be a complete Riemannian manifold and let

$$(8.1) \quad P_g = -h^2 \Delta_g + V, \quad V \in \mathcal{C}^\infty(Y; \mathbb{R}).$$

We make general assumption on  $(Y, g)$  which will allow asymptotically Euclidean and asymptotically hyperbolic infinities.

We assume that  $Y$  is the interior of a compact manifold  $\bar{Y}$  with a  $\mathcal{C}^\infty$  boundary,  $\partial Y \neq \emptyset$ . We choose a defining function of  $\partial Y$ :

$$(8.2) \quad \rho \in \mathcal{C}^\infty(\bar{Y}; [0, \infty)), \quad \{\rho = 0\} = \partial Y, \quad d\rho|_{\partial Y} \neq 0.$$

Let  $p_g = |\xi|_g^2 + V(x)$  be the principal symbol of  $P_g$  and let

$$(x(t), \xi(t)) = \exp tH_{p_g}(x(0), \xi(0)),$$

be the Hamiltonian flow (geodesic flow lifted to  $T^*Y$  when  $V \equiv 0$ ). The first assumption on  $(Y, g)$  we make is a non-trapping (convexity) assumption near infinity formulated using  $\rho$  with properties (8.2):

$$(8.3) \quad \rho(x(t)) \in (0, \epsilon_0), \quad \frac{d}{dt}\rho(x(t)) = 0 \implies \frac{d^2}{dt^2}\rho(x(t)) < 0.$$

The trapped set at energy  $E \in [-\delta, \delta]$  is defined as

$$(x, \xi) \in K_E \iff p_g(x, \xi) = E \text{ and } \exp(\mathbb{R}H_{p_g})(x, \xi) \text{ is compact in } T^*Y.$$

We assume that the trapped set at energies  $|E| \leq \delta$ , (see (1.15)),

$$(8.4) \quad K^\delta \text{ is normally hyperbolic in the sense of (1.17) .}$$

We now make analytic assumptions on  $P$ . For that we first assume that  $P_g$  can be modified inside a compact part of  $Y$ , to obtain an operator

$$P_\infty = -h^2\Delta_g + \tilde{V}, \quad \tilde{V} \in \mathcal{C}^\infty(Y), \quad \tilde{V}|_{\rho < \epsilon_0} = V|_{\rho < \epsilon_0},$$

with the following properties: for some  $s_0 > 0$  and  $C_0 > 0$ ,

$$(8.5) \quad \|\rho^{s_0}(P_\infty - E - i0)^{-1}\rho^{s_0}\|_{L^2(Y) \rightarrow L^2(Y)} \leq \frac{C_0}{h}, \quad |E| \leq \delta,$$

and

$$(8.6) \quad \begin{aligned} u = (P_\infty - E - i0)^{-1}f, \quad f \in \mathcal{C}_c^\infty(Y) &\implies \\ \text{WF}_h(u) \setminus \text{WF}_h(f) &\subset \exp([0, \infty)H_{\text{Re } p_\infty}) \left( \text{WF}_h(f) \cap p_\infty^{-1}(E) \right), \end{aligned}$$

where  $p_\infty \stackrel{\text{def}}{=} |\xi|_g^2 + \tilde{V}$ .

We note that these assumptions do not require that the resolvent of  $P_\infty$  has a meromorphic continuation from  $\text{Im } z > 0$  to the lower half-plane. A stronger conclusion will be possible when we make that assumption: more precisely, for  $\chi \in \mathcal{C}_c^\infty(Y)$ , we assume that the resolvent  $(P_\infty - z)^{-1}$  continues from  $\text{Im } z > 0$  *analytically* to  $[-\delta, \delta] - ih[0, C_0]$ , for some  $C_0 > 0$ , and that for some  $N$ , the following resolvent estimate holds:

$$(8.7) \quad \chi(P_\infty - z)^{-1}\chi = \mathcal{O}_{L^2 \rightarrow L^2}(h^{-N}), \quad z \in [-\delta, \delta] - ih[0, C_0].$$

When  $P_\infty$  is chosen to be selfadjoint, interpolation [7, Lemma 4.7], [36, Lemma 2] shows that (8.7) improves to

$$(8.8) \quad \chi(P_\infty - z)^{-1}\chi = \mathcal{O}_{L^2 \rightarrow L^2}(h^{-1+c_1 \text{Im } z/h} \log(1/h)), \quad z \in [-\delta, \delta] - ih[0, C_0].$$

We can now state a more general version of Theorem 3:

**Theorem 6.** *Suppose that the Riemannian manifold  $(Y, g)$  and the potential  $V$  satisfy the assumptions (8.3), (8.4), (8.5) and (8.6). In particular, the trapped set for the operator  $P = -h^2\Delta_g + V$  is normally hyperbolic.*

*Then, for some constant  $C_1$  (and  $s_0$  in (8.5)), we have*

$$(8.9) \quad \|\rho^{s_0}(P_g - E - i0)^{-1}\rho^{s_0}\|_{L^2(Y) \rightarrow L^2(Y)} \leq C_1 \frac{\log(1/h)}{h}, \quad |E| \leq \delta.$$

*If in addition (8.7) holds then, for any  $\epsilon > 0$ ,  $\chi(P_g - z)^{-1}\chi$  can be continued analytically to  $[-\delta/2, \delta/2] - ih[0, \min(C_0, \lambda_0/2 - \epsilon)]$ , with  $\lambda_0$  given by (1.19), and*

$$(8.10) \quad \chi(P_g - z)^{-1}\chi = \mathcal{O}_{L^2 \rightarrow L^2}(h^{-N}), \quad z \in [-\delta/2, \delta/2] - ih[0, \min(C_0, \lambda_0/2 - \epsilon)],$$

with the improved estimate (1.18) if (8.8) holds.

Before the proof we present two classes of manifolds to which satisfy our assumptions. We say  $(Y, g)$  is *asymptotically Euclidean* if

$$g = \rho^{-4} d\rho^2 + \rho^{-2} g_0(\rho), \quad \text{near } \partial Y,$$

where  $g_0(\rho)$  is a family of metrics on  $\partial Y$  depending smoothly on  $\rho$  up to  $\rho = 0$ . We say  $(Y, g)$  is *evenly asymptotically hyperbolic* if

$$g = \rho^{-2} d\rho^2 + \rho^{-2} g_0(\rho), \quad \text{near } \partial Y,$$

where  $\rho$  is as before but this time  $g_0(\rho)$  is a family of metrics on  $\partial Y$  depending smoothly on  $\rho^2$  (hence *even*) up to  $\rho = 0$ .

In both cases the non-trapping assumption near infinity (8.3) is valid: see [13, Proof of Lemma 4.1] for the asymptotically hyperbolic case; the asymptotically Euclidean case follows from the same proof, with the fourth displayed equation of the proof replaced by [38, (4.3)].

For asymptotically Euclidean manifolds (8.5) and (8.6) follow from the results of [38]. The modification of  $V$  can be done in any way which produces a non-trapping classical flow: for instance we can choose  $\tilde{V} = V + V_{\text{int}}$  where  $V_{\text{int}}$  is a smooth, large non-negative potential (a barrier) supported in  $\{\rho > \epsilon_0\}$ .

To obtain (8.7) more care is needed but, under additional assumptions one can use an adaptation of the method of complex scaling of Aguilar-Combes, Balslev-Combes and Simon – see [41] for the case of manifolds and for references. The simplest example for which this is valid was considered in Theorem 1. For even asymptotically hyperbolic manifolds the properties (8.5), (8.6), and (8.7) all follow from the recent work of Vasy [39].

As long we are not interested in analytic continuation properties, the weaker assumptions (8.5) and (8.6) may hold in the generality considered by Cardoso-Vodev [8].

*Proof of Theorem 6.* To show how Theorem 6 follows from Theorem 2 we use the parametrix construction of [13, §3]. For that we first have to relate the situation in this section to the set-up in Theorem 2. It will be convenient to rescale  $\rho$  so that in (8.3) we can take  $\epsilon_0 = 4$ .

Let  $X$  be any compact manifold without boundary such that  $\bar{Y} \subset X$  is a smooth embedding: for example, we may take  $X$  to be the double of  $\bar{Y}$ . We then extend  $\rho$  to  $\rho \in L^\infty(X)$  to be identically 0 on  $X \setminus Y$ . Let  $P \in \Psi^2(X)$  be any selfadjoint semiclassical differential operator satisfying

$$P|_{\rho>1} = P_g|_{\rho>1}, \quad P = -h^2 \Delta_{g_X} + V_X,$$

where  $g_X$  is a Riemannian metric on  $X$  and  $V_X \in \mathcal{C}^\infty(X; \mathbb{R})$ .

We then take  $W \in \mathcal{C}^\infty(X; [0, \infty))$  such that

$$W(x) = \begin{cases} 0 & \text{for } \rho(x) > 1; \\ 1 & \text{for } \rho(x) < \frac{1}{2}. \end{cases}$$

Let  $\tilde{V} \in \mathcal{C}^\infty(Y)$  be a potential for which (8.5) and (8.6) hold. We notice that one possibility to obtain the required properties for  $P_\infty$  is to take a complex potential  $\tilde{V} = V - iW_\infty$  where,  $W_\infty \in \mathcal{C}^\infty(Y; [0, \infty))$

$$W_\infty(x) = \begin{cases} 0 & \text{for } \rho(x) < 4, \\ 1 & \text{for } \rho(x) > 5, \end{cases}, \quad \text{see Fig. 3.}$$

Using the convexity property (8.3) it is easy to check that this operator satisfies (8.5) and (8.6). Then for  $\text{Im } z > 0$ ,  $|\text{Re } z| \leq \delta$ , define the following holomorphic families of operators

$$R_X(z) = (P - iW - z)^{-1}, \quad R_\infty(z) = (P_\infty - z)^{-1}.$$

Due to the compactness of  $X$ , the family of operators  $R_X(z): L^2(X) \rightarrow L^2(X)$  is meromorphic for  $z \in \mathbb{C}$ . The resolvent  $R_X(z)$  is estimated in Theorem 2. For the moment we only assume that  $R_\infty(z): L^2(Y) \rightarrow L^2(Y)$  is holomorphic for  $\text{Im } z > 0$  and satisfies (8.5), (8.6).

Now take a cutoff function  $\chi_X \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$  with

$$\text{supp } \chi_X \subset (2, \infty), \quad \text{supp}(1 - \chi_X) \subset (-\infty, 3).$$

We put and  $\chi_\infty = 1 - \chi_X$ .

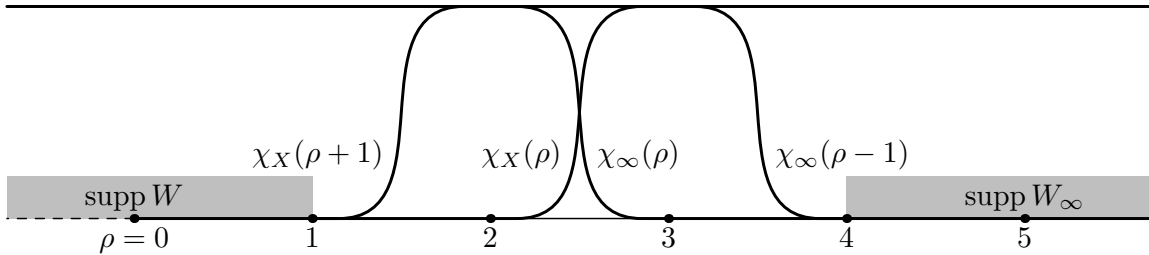


FIGURE 3. Schematic representation of the cut-offs used in the proof of Theorem 6 as functions of  $\rho(x)$ . The spatial infinity is represented by  $\rho(x) = 0$  and  $X \setminus Y$  corresponds to  $\rho(x) \leq 0$ .

Our first Ansatz for the inverse of  $(P_g - z)$  is the operator

$$F(z) = \chi_X(\rho(\bullet) + 1)R_X(z)\chi_X(\rho(\bullet)) + \chi_\infty(\rho(\bullet) - 1)R_\infty(z)\chi_\infty(\rho(\bullet)).$$

Note that  $F(z): L^2(Y) \rightarrow L^2(Y)$  for  $\text{Im } z > 0$  since all the cut-off functions are supported away from  $X \setminus Y$ . Also, the support properties of  $W$ ,  $W_\infty$  and  $\chi_X$  show that

$$\begin{aligned} (P_g - z)\chi_X(\rho(\bullet) + 1) &= \chi_X(\rho(\bullet) + 1)(P - iW - z) + [\chi_X(\rho(\bullet) + 1), h^2\Delta_g], \\ (P_g - z)\chi_\infty(\rho(\bullet) - 1) &= \chi_\infty(\rho(\bullet) - 1)(P_\infty - z) + [\chi_\infty(\rho(\bullet) - 1), h^2\Delta_g]. \end{aligned}$$

Hence

$$(P_g - z)F(z) = I + A_X(z) + A_\infty(z),$$

where

$$\begin{aligned} A_X(z) &= [\chi_X(\rho(\bullet) + 1), h^2 \Delta_g] R_X(z) \chi_X(\rho(\bullet)), \\ A_\infty(z) &= [\chi_\infty(\rho(\bullet) - 1), h^2 \Delta_g] R_\infty(z) \chi_\infty(\rho(\bullet)). \end{aligned}$$

Note that  $A_X(z)^2 = A_\infty(z)^2 = 0$ , due to the support properties

$$\begin{aligned} (8.11) \quad & \text{supp } d(\chi_X(\rho(\bullet) + 1)) \cap \text{supp } \chi_X(\rho(\bullet)) = \emptyset, \\ & \text{supp } d(\chi_\infty(\rho(\bullet) - 1)) \cap \text{supp } \chi_\infty(\rho(\bullet)) = \emptyset. \end{aligned}$$

Moreover, thanks to assumptions (8.3) and (8.6) (see [13, Lemma 3.1]),

$$(8.12) \quad \|A_\infty(z) A_X(z)\|_{L^2(Y) \rightarrow L^2(Y)} = \mathcal{O}(h^\infty), \text{ uniformly for } \text{Im } z > 0, |\text{Re } z| \leq \delta.$$

Consequently

$$(8.13) \quad (P_g - z)F(z)((I - A_X(z) - A_\infty(z) + A_X(z)A_\infty(z)) = I - E(z),$$

$$(8.14) \quad \text{where } E(z) = A_\infty(z)A_X(z) - A_\infty(z)A_X(z)A_\infty(z).$$

Using (8.12) we see that  $E(z) = \mathcal{O}(h^\infty)_{L^2(Y) \rightarrow L^2(Y)}$ , uniformly for  $\text{Im } z > 0, |\text{Re } z| \leq \delta$ . This allows to write an explicit expression for  $(P_g - z)^{-1}$ :

$$(P_g - z)^{-1} = F(z)(I - A_X(z) - A_\infty(z) + A_X(z)A_\infty(z)) \sum_{n=0}^{\infty} E(z)^n.$$

We now want to estimate  $\|\rho^{s_0}(P_g - z)^{-1}\rho^{s_0}\|_{L^2(Y) \rightarrow L^2(Y)}$ . For this we expand the above identity using the expression of  $F(z)$  (some terms vanish due to the support properties (8.11)). Denoting  $a_X = \|R_X(z)\|$ ,  $a_\infty = \|\rho^{s_0} R_\infty(z) \rho^{s_0}\|$ , we get the bound

$$(8.15) \quad \|\rho^{s_0}(P_g - z)^{-1}\rho^{s_0}\| \leq C(a_\infty + a_X + ha_\infty a_X + h^2 a_\infty^2 a_X) + \mathcal{O}(h^\infty).$$

Finally, we use the bounds (8.5) for  $a_\infty$ , the bound (1.18) for  $a_X$  (with  $\text{Im } z \geq 0$ ), and obtain the desired estimate (8.9).

When the assumption (8.7) holds, the construction shows that for  $\chi \in \mathcal{C}_c^\infty(Y)$  equal to 1 on a sufficiently large set,

$$\chi(P_g - z)^{-1}\chi = \chi F(z)\chi(I - A_X(z) - A_\infty(z) + A_X(z)A_\infty(z))\chi \sum_{n=0}^{\infty} (E(z)\chi)^n,$$

continues analytically to the same region as *both*  $R_X(z)$  and  $\chi R_\infty(z)\chi$ . The same expansion as above allows to bound from above  $\|\chi(P_g - z)^{-1}\chi\|$  by the same expression as in (8.15), now using  $a_X = \|\chi R_X(z)\chi\|$ ,  $a_\infty = \|\chi R_\infty(z)\chi\|$ . By using (1.18) for  $a_X$ , resp. (8.7) for  $a_\infty$  (with now  $\text{Im } z$  taking negative values), we obtain (8.10).  $\square$

For completeness we conclude this section with the proof of Theorem 1. The conclusion is valid under more general assumptions of Theorem 6.

*Proof of Theorem 1.* In the notation of Theorem 6, (1.3) is equivalent to the estimate

$$(8.16) \quad \|\chi\psi(P_g)e^{-itP_g/h}\chi\|_{L^2(Y)\rightarrow L^2(Y)} \leq C \frac{\log 1/h}{h^{1+c_0\gamma}} e^{-\gamma t} + \mathcal{O}(h^\infty), \quad \gamma = \frac{1}{2}(\lambda_0 - \epsilon),$$

valid (with different constants) for any  $\chi \in \mathcal{C}_c^\infty(Y)$ . Let  $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{C})$  be an almost analytic extension of  $\psi$ , that is a function with the property that  $\tilde{\psi}|_{\mathbb{R}} = \psi$  and  $\bar{\partial}_z \tilde{\psi}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$  (see for instance [43, Theorem 3.6]). We can construct  $\tilde{\psi}$  so that  $\operatorname{supp} \tilde{\psi} \subset [-\delta/2, \delta/2] - i[-\delta, \delta]$ . We start with Stone's formula

$$\chi\psi(P_g)e^{-itP_g/h}\chi = \frac{1}{2\pi i} \int_{\mathbb{R}} \psi(\lambda) e^{-i\lambda t} \chi \left( (P_g - \lambda - i0)^{-1} - (P_g - \lambda + i0)^{-1} \right) \chi d\lambda.$$

We now write  $R_-(z) = (P_g - z)^{-1}$ , for the resolvent in  $\operatorname{Im} z < 0$  (that is for the analytic continuation of  $(P_g - (z - i0))^{-1}$  from  $\operatorname{Im} z < 0$ ) and  $R_+(z)$  for the meromorphic continuation of the resolvent from  $\operatorname{Im} z > 0$  to the lower half-plane. We then apply Green's formula to obtain, for  $0 \leq \gamma < \lambda_0$ ,

$$(8.17) \quad \begin{aligned} \chi\psi(P_g)e^{-itP_g/h}\chi &= \frac{1}{2\pi i} \int_{\operatorname{Im} z = -\gamma h} e^{-itz/h} \chi(R_+(z) - R_-(z)) \chi \tilde{\psi}(z) dz \\ &\quad + \frac{1}{\pi} \iint_{-\gamma h \leq \operatorname{Im} z \leq 0} e^{-itz/h} \chi(R_+(z) - R_-(z)) \chi \bar{\partial}_z \tilde{\psi}(z) dm(z), \end{aligned}$$

where  $dm(z)$  is the Lebesgue measure on  $\mathbb{C}$ . From (1.18) (see Theorem 6) we get

$$\|\chi R_+(z)\chi\|_{L^2 \rightarrow L^2} \leq Ch^{-(1+c_0\gamma)} \log(1/h), \quad \|\chi R_-(z)\chi\|_{L^2 \rightarrow L^2} \leq C/|\operatorname{Im} z|,$$

for  $-\gamma h \leq \operatorname{Im} z \leq 0$ . Inserting these bounds in (8.17) gives (8.16) and that proves (a generalized version of) Theorem 1.  $\square$

## 9. DECAY OF CORRELATIONS FOR CONTACT ANOSOV FLOWS: PROOF OF THEOREM 5

Most of this section is devoted to the proof of Theorem 4. This proof will be obtained by adapting the proof of Theorem 2, after reviewing the geometric point of view of Tsujii [37] and Faure-Sjöstrand [21] (see also [12]). At the end of the section we deduce Theorem 5 on the decay of correlations.

**9.1. Geometric structure.** Let  $X$  be a smooth compact manifold of dimension  $d = 2k - 1$ ,  $k \geq 2$ . We assume that  $X$  is equipped with a contact 1-form  $\alpha$ , that is, a form such that  $(d\alpha)^{\wedge k-1} \wedge \alpha$  is non-degenerate. The Reeb vector field,  $\Xi$ , is defined as the unique vector field satisfying

$$\Xi_x \in \ker \alpha_x, \quad \alpha_x(\Xi_x) = 1, \quad x \in X.$$

We assume that

$$(9.1) \quad \gamma_t \stackrel{\text{def}}{=} \exp t\Xi \text{ defines an Anosov flow on } X.$$



That means that at each point  $x \in X$ , the tangent space has a  $\gamma_t$ -invariant decomposition into neutral (one dimensional), stable and unstable subspaces (each  $(k-1)$ -dimensional):

$$(9.2) \quad T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x), \quad E_0(x) = \mathbb{R}\Xi_x.$$

We note that  $E_u(x) \oplus E_s(x)$  span the kernel of  $\ker \alpha_x$ .

The dual decomposition is obtained by taking  $E_0^*(x)$  to be the annihilator of  $E_s(x) \oplus E_u(x)$ ,  $E_u^*(x)$  the annihilator of  $E_u(x) \oplus E_0(x)$ , and similarly for  $E_s^*(x)$ . That makes  $E_s^*(x)$  dual to  $E_u(x)$ ,  $E_u^*(x)$  dual to  $E_s(x)$ , and  $E_0^*(x)$  dual to  $E_0(x)$ . The fiber of the cotangent bundle then decomposes as

$$(9.3) \quad T_x^* X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x).$$

The distributions  $E_s^*(x)$  and  $E_u^*(x)$  have only Hölder regularity, but  $E_0^*(x)$  and  $E_s^*(x) \oplus E_u^*(x)$  are smooth, and  $E_0^*(x) = \mathbb{R}\alpha_x \subset T_x^* X$ .

The approach of [21] highlights the analogy between this dynamical setting and the scattering theory for the Schrödinger equation. The role of the Schrödinger operator is played by the (rescaled) generator of the flow  $\gamma_t = \exp t\Xi$ :

$$(9.4) \quad \gamma_t^* u = e^{itP/h} u, \quad u \in \mathcal{C}^\infty(X), \quad P = -ih\Xi.$$

The principal symbol of  $P$  simply reads  $p(x, \xi) = \xi(\Xi_x)$ .

The flow  $\gamma_t$  can be lifted to a Hamiltonian flow  $\varphi_t$  on  $T^*X$ :

$$\varphi_t : (x, \xi) \longmapsto (\gamma_t(x), {}^t d\gamma_t(x)^{-1} \xi),$$

which is generated by  $p(x, \xi)$ :  $\varphi_t = \exp tH_p$ .

For each energy  $E \in \mathbb{R}$ , the energy shell  $p^{-1}(E)$  is a union of affine subspaces:

$$p^{-1}(E) = \bigcup_{x \in X} \{(x, \xi) : \alpha_x(\xi) = E\} = \bigcup_{x \in X} (E\alpha_x + E_u^*(x) + E_s^*(x)).$$

We note that each of these energy shells has infinite volume; as opposed to the scattering theory setting, infinity occurs here in the momentum direction (the fibers), while the spatial direction is compact.

The Anosov assumption implies that for  $t > 0$ ,

$$(9.5) \quad |\varphi_t(x, \xi)| \leq C e^{-\lambda t} |\xi|, \quad \xi \in E_s^*(x), \quad |\varphi_{-t}(x, \xi)| \leq C e^{-\lambda t} |\xi|, \quad \xi \in E_u^*(x),$$

where  $|\bullet| = |\bullet|_y$  denotes a norm on  $T_y^* X$ , and we consider  $\varphi_t(x, \xi) \in T_{\pi(\varphi_t(x, \xi))}^* X$ . Since the action of  $\varphi_t$  inside each fiber  $T_x^* X$  is linear, we see that the only trapped point in  $T^* X$  must be on the line  $E_0^*(x)$ . More precisely, the trapped set at energy  $E \in \mathbb{R}$  is given by

$$K_E = \bigcup_{x \in X} (E_0^*(x) \cap p^{-1}(E)) = \bigcup_{x \in X} E\alpha_x,$$

that is  $K_E$  is the image of the section  $E\alpha$  in  $T^* X$ .

Stacking together energies  $E \in (1 - \delta, 1 + \delta)$ ,  $0 < \delta < 1$ , we obtain the trapped set

$$K^\delta = K = \bigcup_{|E-1|<\delta} K_E = \{E\alpha_x, x \in X, |E-1| < \delta\} \subset T^*X.$$

This trapped set is normally hyperbolic in the sense of (1.17).

Indeed, we first note that  $K^\delta$  is a symplectic submanifold of  $T^*X$  of dimension  $d+1 = 2k$ . Indeed, using  $(x, E)$ ,  $x \in X$ , as coordinates on  $K^\delta$ ,  $(x, E) \mapsto E\alpha_x$ , we have

$$\omega|_{K^\delta} = d(E\alpha) = dE \wedge \alpha + E d\alpha.$$

This form is nondegenerate for  $E \neq 0$  since  $\alpha$  is a contact form.

The tangent space to  $K^\delta$  is given by the image of the differential of

$$X \times \mathbb{R} \ni (x, E) \mapsto E\alpha_x \stackrel{\text{def}}{=} (x, \xi = E\beta(x)),$$

where we see  $\beta(x)$  as the vector in  $\mathbb{R}^d$  representing  $\alpha_x$ . Hence,

$$(9.6) \quad \begin{aligned} T_{E\alpha_x} K^\delta &= E(d\alpha)_x(T_x X, \bullet) + \mathbb{R}\alpha_x = \{(v, E d\beta(x)v + s\beta(x)) : v \in T_x X, s \in \mathbb{R}\} \\ &\subset T_x X \oplus T_x^* X \equiv T_{E\alpha_x}(T^* X). \end{aligned}$$

Here  $d\beta(x)$  can be interpreted as the Jacobian matrix  $\partial\beta/\partial x$  on  $\mathbb{R}^{d-1}$ .

For each  $x \in X$ , the symplectic orthogonal to  $T_{E\alpha_x} K^\delta$ , denoted  $(T_{E\alpha_x} K^\delta)^\perp$ , can be obtained by lifting the vectors in  $\ker \alpha_x$  as follows:

$$v \in \ker \alpha_x \mapsto L_E^\perp(v) \stackrel{\text{def}}{=} (v, E {}^t(d\beta(x))v) \in T_x X \oplus T_x^* X \equiv T_{E\alpha_x}(T^* X),$$

where  ${}^t(d\beta(x))$  denotes the transpose of  $d\beta(x)$ . This subspace  $(T_{E\alpha_x} K^\delta)^\perp$  is symplectic and transverse to  $K^\delta$ :

$$T_\rho K^\delta \oplus (T_\rho K^\delta)^\perp = T_\rho(T^* X), \quad \forall \rho = E\alpha_x \in K^\delta.$$

Since  $\ker \alpha_x = E_u(x) \oplus E_s(x)$ , we can naturally split the orthogonal subspace into

$$(T_\rho K^\delta)^\perp = E_\rho^+ \oplus E_\rho^-, \quad E_\rho^+ = L_E^\perp(E_u(x)), \quad E_\rho^- = L_E^\perp(E_s(x)), \quad \rho = E\alpha_x \in K^\delta.$$

The distributions  $E_{E\alpha_x}^\pm$  are transverse to each other and flow-invariant.  $E_{E\alpha_x}^+$  is a particular subspace of the global unstable subspace  $E_u(x) \oplus E_u^*(x) \subset T_{E\alpha_x}(T^* X)$ , and similarly for  $E_{E\alpha_x}^-$ . Hence, in the present setting, the subspaces  $E_\rho^\pm$  exactly correspond to the subspaces described in Lemma 4.1.

**9.2. Microlocally weighted spaces and the definition of resonances.** Following [12] we now review the construction [21] of Hilbert spaces on which  $P - z$  (with  $P$  given in (9.4)) is a Fredholm operator for  $\text{Im } z > -\beta h$ , for some arbitrary  $\beta > 0$ .

The key to the definition of these Hilbert spaces is a construction of a weight function which we quote from [21, Lemma 1.2] and [12, Lemma 3.1]. We use the notation  $E_\bullet^* = \bigcup_{x \in X} E_\bullet^*(x) \subset T^* X$ .

**Lemma 9.1.** *Let  $U_0, U'_0$  be conic neighbourhoods of  $E_0^*$ , with  $U_0 \Subset U'_0$  and  $U'_0 \cap (E_u^* \cup E_s^*) = \emptyset$ . There exist real-valued functions  $m \in S^0(T^*X)$ ,  $f_0 \in S^1(T^*X)$  such that*

- (1)  *$m$  is positively homogeneous of degree 0 for  $|\xi| \geq 1/2$ , equal to  $-1, 0, 1$  near the intersection of  $\{|\xi| \geq 1/2\}$  with  $E_u^*, E_0^*, E_s^*$ , respectively, and*

$$(9.7) \quad H_p m < 0 \text{ near } (U'_0 \setminus U_0) \cap \{|\xi| > 1/2\}, \quad H_p m \leq 0 \text{ on } \{|\xi| > 1/2\};$$

- (2)  *$\langle \xi \rangle^{-1} f_0 \geq c > 0$  for some constant  $c$ ;*

- (3) *the function  $G = m \log f_0$  satisfies*

$$(9.8) \quad H_p G \leq -2 \text{ on } \{|\xi| \geq 1/2\} \setminus U_0, \quad H_p G \leq 0 \text{ on } \{|\xi| \geq 1/2\}.$$

The function  $G$  also satisfies derivative bounds

$$(9.9) \quad G = \mathcal{O}(\log \langle \xi \rangle), \quad \partial_x^\alpha \partial_\xi^\beta H_p^k G = \mathcal{O}(\langle \xi \rangle^{-|\beta|+\epsilon}), \quad |\alpha| + |\beta| + k \geq 1,$$

for any  $\epsilon > 0$ .

As in [12, §3] we define

$$(9.10) \quad H_{tG}(X) \stackrel{\text{def}}{=} e^{-tG^w} L^2(X, dx),$$

where  $t > 0$  is a positive parameter.

The domain of  $P$  acting on  $H_{tG}$  is defined as

$$(9.11) \quad \mathcal{D}_{tG} \stackrel{\text{def}}{=} \{u \in \mathcal{D}'(X) : u, Pu \in H_{tG}\}.$$

The action of  $P$  on  $H_{tG}$  is equivalent to the action of the operator  $P_{tG}$  on  $L^2$ :

$$(9.12) \quad \begin{aligned} P_{tG} &\stackrel{\text{def}}{=} e^{tG^w} P e^{-tG^w} = \exp(t \operatorname{ad}_{G^w}) P \\ &= \sum_{k=0}^N \frac{t^k}{k!} \operatorname{ad}_{G^w}^k P + R_N(x, hD), \quad R_N \in h^{N+1} S^{-N+\epsilon}, \quad \forall \epsilon > 0. \end{aligned}$$

The validity of (9.12) follows from the fact that the operators  $e^{\pm tG^w}$  are pseudodifferential operators [43, Theorem 8.6], hence the pseudodifferential calculus applies directly [43, Theorem 9.5, Theorem 14.1]. This expansion and the arguments in [21, §3] give

**Proposition 9.2.** *For  $P_{tG}$  defined by (9.12), we have*

i) *the operator  $P_{tG} - z : \mathcal{D}(P_{tG}) \rightarrow L^2$  is Fredholm of index zero for  $\operatorname{Im} z > -th$ . Here  $\mathcal{D}(P_{tG})$  is the domain of  $P_{tG}$ .*

ii) *for  $C > 0$  large enough,  $(P_{tG} - z)$  is invertible on  $\{\operatorname{Im} z > Ch\}$ .*

In [21] the above construction was performed, replacing the  $h$ -quantization by the  $h = 1$  quantization. It lead to the construction of  $H_{tG,1}(X) = e^{-tG^w(x,D)} L^2(X)$  equal, as vector space, to the above  $h$ -dependent space  $H_{tG}(X)$ . The norms of these two spaces are equivalent with one another, but in an  $h$ -dependent way:

$$(9.13) \quad h^N \|u\|_{H_{tG,1}(X)} / C_0 \leq \|u\|_{H_{tG}} \leq C_0 h^{-N} \|u\|_{H_{tG,1}(X)},$$

– see [12, §3]. As a consequence, if we call  $P_1 = -i\Xi$ , Theorem 4 translates into the fact that  $P_{tG,1} \stackrel{\text{def}}{=} e^{tG^w(x,D)} P_1 e^{-tG^w(x,D)}$  is Fredholm in the strip  $\{\text{Im } \lambda > -t\}$ , admits finitely many eigenvalues in the strip  $\{\text{Im } \lambda \geq -\lambda/2 + \epsilon\}$ , and satisfies the resolvent estimate

$$(9.14) \quad \|(P_{tG,1} - \lambda)^{-1}\|_{L^2 \rightarrow L^2} = \mathcal{O}(\lambda^{N_0}), \quad \text{Im } \lambda \geq -\lambda/2 + \epsilon, \quad |\text{Re } \lambda| \geq C.$$

**9.3. Reduction to Theorem 2.** In order to prove Theorem 4, we proceed as in the proof of Theorem 6 in §8, by constructing two operators which microlocally agree with  $P_{tG}$  (up to negligible error terms) on different subsets of  $T^*X$ .

Let  $W \in \Psi^1(X)$  be as in Example 2 of §1.3. The trapped set defined in §1.3 agrees with the trapped set in §9.1 and, as shown in (9.5), it satisfies, the assumptions of normal hyperbolicity. Hence Theorem 2 applies to  $\tilde{P} = P - iW$ . If  $A \in \Psi^{\text{comp}}(X)$  satisfies

$$(9.15) \quad \text{WF}_h(A) \Subset (G^{-1}(0))^\circ, \quad \text{WF}_h(A) \cap \{(x, \xi) : |\xi|_{g_x} \geq M\} = \emptyset,$$

(where  $M$  is the one appearing in the definition of  $W$  – see (1.13)) then

$$(9.16) \quad A\tilde{P} = AP_{tG} + \mathcal{O}(h^\infty)_{\mathcal{D}' \rightarrow \mathcal{C}^\infty}.$$

We now introduce an operator which has better global properties and agrees with  $P_{tG}$  near infinity. For that we proceed as in the proof of Theorem 6, and take  $W_\infty \in \Psi^{\text{comp}}(X)$  such that

$$(9.17) \quad \text{WF}_h(W_\infty) \Subset (G^{-1}(0))^\circ, \quad \text{WF}_h(I - W_\infty) \subset \mathbb{C}K^\delta, \quad \text{WF}_h(W) \cap \text{WF}_h(W_\infty) = \emptyset.$$

We then put

$$P_\infty = P_{tG} - iW_\infty.$$

Then for any  $B$  with  $\text{WF}_h(I - B) \subset \mathbb{C}K^\delta$ ,

$$(9.18) \quad (I - B)P_\infty = (I - B)P_{tG} + \mathcal{O}(h^\infty)_{\mathcal{D}' \rightarrow \mathcal{C}^\infty}.$$

Properties of the operator  $P_\infty$  are listed in the following

**Lemma 9.3.** *Fix  $\beta > 0$  and let  $t$  be large enough so that  $P_{tG} - z$  is a Fredholm operator for  $\text{Im } z > -\beta h$ . Then, there exists  $N_0$  and  $h_0$  such that, for  $0 < h < h_0$ ,*

$$\|(P_\infty - z)^{-1}\|_{L^2 \rightarrow L^2} \leq h^{-N_0}, \quad z \in [1 - \delta/2, 1 + \delta/2] - ih[0, \beta].$$

*In addition the analogue of (8.6) holds for  $P_\infty$ : in the same range of  $z$ ,*

$$(9.19) \quad u = (P_\infty - z)^{-1}f, \quad f \in \mathcal{C}^\infty(X) \implies \text{WF}_h(u) \setminus \text{WF}_h(f) \subset \exp([0, \infty)H_p) (\text{WF}_h(f) \cap p^{-1}(\text{Re } z)).$$

*Proof.* The first part follows from the now standard non-trapping estimates (see [35, §4]). In the setting of Anosov flows the details are presented in the proof of [12, Lemma 5.1] (only the escape function constructed in Lemma 4.5 above is needed).

The propagation result is a real principal type propagation result [43, Theorem 12.5] which holds when the imaginary part of the symbol is non-positive – see Lemma A.1 below for a dynamical version.  $\square$

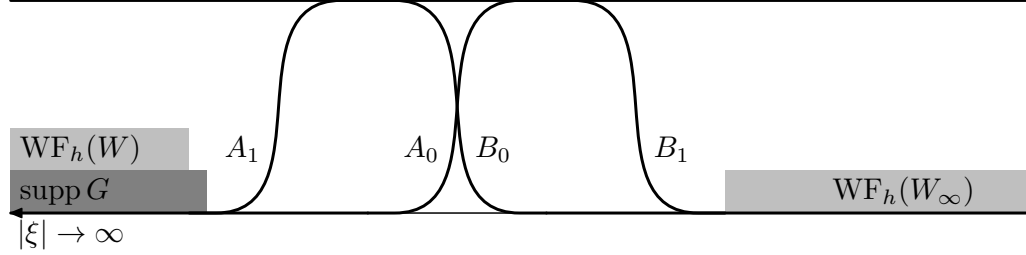


FIGURE 4. Schematic representation of pseudodifferential cut-offs used in the proof of Theorem 4. The horizontal axis corresponds to  $|\xi|$ , the cotangent variable. Infinity in  $|\xi|$  plays the role of  $\rho = 0$  in Fig. 3. The asymmetry is intentional, to stress that there is no need for an auxiliary manifold, as opposed to the proof of Theorem 6 illustrated in Fig. 3.

*Proof of Theorem 4.* The proof is a repetition of the proof of Theorem 3 with  $R_X$  replaced by  $(\tilde{P} - z)^{-1}$  and  $R_\infty$  by  $(P_\infty - z)^{-1}$ . The spatial cut-off functions are replaced by pseudodifferential operators:  $\chi_X(\rho(x))$  is replaced by  $A_0 \in \Psi^{\text{comp}}(X)$ , satisfying

$$\text{WF}_h(A_0) \cap \{(x, \xi) : |\xi|_{g_x} \geq M\} = \emptyset, \quad \text{WF}_h(I - A_0) \cap \text{WF}_h(W_\infty) = \emptyset,$$

where  $M$  is given in the definition of  $W$ , see (1.13). The function  $\chi_X(\rho(x) + 1)$  is replaced by  $A_1 \in \Psi^{\text{comp}}(X)$ , where

$$\text{WF}_h(I - A_1) \cap \text{WF}_h(A_0) = \emptyset, \quad \text{WF}_h(A_1) \cap \{(x, \xi) : |\xi|_{g_x} \geq M\} = \emptyset,$$

$\chi_\infty(\rho(x))$  is replaced by  $B_0 \stackrel{\text{def}}{=} I - A_0 \in \Psi^0(X)$ , and finally  $\chi_\infty(\rho(x) - 1)$  by  $B_1 \in \Psi^0(X)$ , where

$$\text{WF}_h(W_\infty) \cap \text{WF}_h(B_1) = \emptyset, \quad \text{WF}_h(I - B_1) \cap \text{WF}_h(B_0) = \emptyset.$$

We also require that

$$\text{WF}_h(A_1), \text{WF}_h(I - B_1) \subset (G^{-1}(0))^\circ.$$

The parametrix is now obtained by putting

$$F(z) = A_1(P - iW - z)^{-1}A_0 + B_1(P_{tG} - iW_\infty - z)^{-1}B_0.$$

Using (9.16), (9.18) and Lemma 9.3 we obtain the theorem by proceeding as in the proof of Theorem 3 in §8.  $\square$

*Proof of Theorem 5.* We will use the nonsemiclassical operator  $P_1 = -i\Xi$ . It is selfadjoint on  $L^2(X)$  – see [21, Appendix A] – hence, by Stone’s formula, we get for any  $f, g \in \mathcal{C}^\infty(X)$

$$\begin{aligned} \int_X \gamma_{-t}^* f g \, dx &= \langle e^{-itP_1} f, \bar{g} \rangle \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} \left( \langle (P_1 - \lambda - i0)^{-1} f, \bar{g} \rangle - \langle (P_1 - \lambda + i0)^{-1} f, \bar{g} \rangle \right) d\lambda \\ &= \frac{1}{2\pi i} \sum_{\pm} \mp \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda \pm i0)^{-1} (P_1 + i)^k f, \bar{g} \rangle d\lambda. \end{aligned}$$

Here the brackets  $\langle \bullet, \bullet \rangle$  represent  $L^2(X)$  scalar products.

For  $t > 0$  we can deform the contour in the integral corresponding to  $+i0$  ( $\lambda$  approaching the real axis from below), where  $\|(P_1 - \lambda)^{-1}\| \leq |\operatorname{Im} \lambda|^{-1}$ , so that for  $k > 1$  the integral is bounded as

$$-\frac{1}{2\pi i} \int_{\mathbb{R}-iA} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda)^{-1} (P_1 + i)^k f, \bar{g} \rangle d\lambda = \mathcal{O}(e^{-tA} \|f\|_{H^k} \|g\|_{L^2}).$$

Thus,

$$\int_X \gamma_t^* f g \, dx = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda - i0)^{-1} (P_1 + i)^k f, \bar{g} \rangle + \mathcal{O}_{f,g}(e^{-tA}),$$

for any  $A$ , with the bounds depending on seminorms of  $f$  and  $g$  in  $\mathcal{C}^\infty$ . We now use the nonsemiclassical weights  $G^w(x, D)$  constructed in §9.2 to conjugate  $P_1$ , and write

$$\begin{aligned} \int_X \gamma_t^* f g \, dx &= \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_{tG,1} - \lambda - i0)^{-1} (P_{tG,1} + i)^k e^{tG^w(x,D)} f, e^{-tG^w(x,D)} \bar{g} \rangle + \mathcal{O}_{f,g}(e^{-tA}). \end{aligned}$$

The nonsemiclassical analogue (9.14) of Theorem 4 shows that, by taking  $k > N_0 + 1$ , we may deform the contour of integration down to  $\operatorname{Im} \lambda = -\lambda_0/2 + \epsilon$ , collecting finitely many poles  $\mu_j$ , to finally obtain the expansion (1.22).  $\square$

## APPENDIX: EVOLUTION FOR THE CAP-MODIFIED HAMILTONIAN.

In the appendix we show some properties of the CAP-modified Hamiltonian, that is the Hamiltonian modified by adding a complex absorbing potential. At first we work under the general assumptions (1.9).

The semigroup  $\exp(-it(P - iW)/h) : L^2(X) \rightarrow L^2(X)$  is defined using the Hille-Yosida theorem: for  $h$  small  $P - iW - i$  is invertible as its symbol is elliptic in the semiclassical sense (see (1.11) and [43, Theorem 4.29]). Ellipticity assumption for large values of  $\xi$  also

shows that  $P - iW$  is a Fredholm operator, and the comment about invertibility shows that it has index 0. The estimate

$$\|(P - iW - z)u\| \|u\| \geq -\operatorname{Im} \langle (P - iW - z)u, u \rangle \geq \operatorname{Im} z \|u\|^2, \quad u \in H_h^m(X),$$

then shows invertibility for  $\operatorname{Im} z > 0$ , with the bound

$$\|(P - iW - z)^{-1}\|_{L^2 \rightarrow L^2} \leq \frac{1}{\operatorname{Im} z}, \quad \operatorname{Im} z > 0.$$

Since the domain of  $P - iW$  is given by  $H^m(X)$  which is dense in  $L^2$ , the hypotheses of the Hille-Yosida theorem are satisfied, and

$$(A.1) \quad \begin{aligned} \|e^{-it(P-iW)/h}\|_{L^2 \rightarrow L^2} &\leq 1, \quad t \geq 0, \\ e^{-it(P-iW)/h} e^{-is(P-iW)/h} &= e^{-i(t+s)(P-iW)/h}, \quad t, s \geq 0. \end{aligned}$$

Alternatively we can show the existence of the semigroup  $\exp(-it(P-iW)/h)$  using energy estimates, just as is done in the proof of [43, Theorem 10.3]. We get that for any  $T > 0$ ,

$$(A.2) \quad e^{-it(P-iW)/h} \in C([0, T]; \mathcal{L}(H_h^s(X), H_h^s(X))) \cap C^1([0, T]; \mathcal{L}(H_h^s, H_h^{s-m})).$$

Our final estimates will all be given for  $L^2$  only and that is sufficient for our purposes.

The first result we state concerns propagation of semiclassical wave front sets. We recall the notation  $\varphi_t = \exp(tH_p)$  for the Hamiltonian flow generated by  $p(x, \xi)$ .

**Lemma A.1.** *Suppose that  $A \in \Psi^{\operatorname{comp}}(X)$ . Then for any  $T$  independent of  $h$  there exists a smooth family of operators*

$$(A.3) \quad [0, T] \ni t \longmapsto Q(t) \in \Psi^{\operatorname{comp}}(X), \quad \operatorname{WF}_h(I - Q(t)) \cap \varphi_t(\operatorname{WF}_h(A)) = \emptyset,$$

such that

$$(A.4) \quad (I - Q(t)) e^{-it(P-iW)/h} A = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

In addition if  $\operatorname{WF}_h(A) \subset w^{-1}([\epsilon, \infty))$ ,  $\epsilon > 0$ , then for any fixed  $t > 0$ ,

$$(A.5) \quad e^{-it(P-iW)/h} A = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}, \quad A e^{-it(P-iW)/h} = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

*Proof.* We first construct  $Q(t)$  using a semiclassical adaptation of a standard microlocal procedure – see [27, §23.1]. For that, let  $Q(0) \in \Psi^{\operatorname{comp}}(X)$  be an operator satisfying  $\operatorname{WF}_h(I - Q(0)) \cap \operatorname{WF}_h(A) = \emptyset$ , and with the principal symbol,  $q_0(0)$ , independent of  $h$ . Using the fact that the flow  $\varphi_t$  is defined for all  $t$  we put  $q_0(t) \stackrel{\text{def}}{=} \varphi_{-t}^* q_0(0)$ . In terms of the Poisson bracket on the extended phase space  $T^*(\mathbb{R} \times X) \ni (t, x, \tau, \xi)$ , this means that the function  $q_0(t)$  satisfies the identity  $\{\tau + p, q_0(t)\} = 0$ . Consequently, at the quantum level we have

$$\begin{aligned} [hD_t + P, \operatorname{Op}_h^w(q_0(t))] &= hR_1(t), \quad R_1(t) \in \Psi^{\operatorname{comp}}(X), \\ \operatorname{Op}_h^w(q_0(0)) - Q(0) &= hE_1, \quad E_1 \in \Psi^{\operatorname{comp}}(X), \end{aligned}$$

and the principal symbols of  $R_1, E_1, r_1, e_1 \in \mathcal{C}_c^\infty(T^*X)$ , are independent of  $h$ . If  $p_1 = \sigma((P - \text{Op}_h^w(p))/h)$ , we then solve (in the unknown  $q_1(t)$ ) the equation

$$\{\tau + p, q_1(t)\} = r_1 - \{p_1, q_0(t)\}, \quad q_1(0) = e_1.$$

By iteration of this procedure we obtain  $q_\ell \in \mathcal{C}^\infty(T^*X)$  such that

$$\begin{aligned} [hD_t + P, \sum_{\ell=0}^{N-1} h^\ell \text{Op}_h^w(q_\ell(t))] &= h^N R_N(t), \quad R_N(t) \in \Psi^{\text{comp}}(X), \\ \sum_{\ell=1}^{N-1} h^\ell \text{Op}_h^w(q_\ell(0)) - Q(0) &= h^N E_N, \quad E_N \in \Psi^{\text{comp}}(X). \end{aligned}$$

By a standard Borel resummation we may construct  $Q(t) \in \Psi^{\text{comp}}(X)$  such that  $Q(t) \sim \sum_{\ell \geq 0} h^\ell \text{Op}_h^w(q_\ell(t))$ .

For any  $N > 0$  we can iteratively construct a sequence of auxiliary operators  $Q_j(t) = Q_j(t)^* \in \Psi^{\text{comp}}(X)$ ,  $0 \leq j \leq N$ , satisfying

$$\begin{aligned} \text{WF}_h(I - Q_{j+1}(t)) \cap \text{WF}_h(Q_j(t)) &= \text{WF}_h(I - Q_j(t)) \cap \varphi_t(\text{WF}_h(A)) \\ &= \text{WF}_h(I - Q(t)) \cap \text{WF}_h(Q_j(t)) = \emptyset, \\ [Q_j(t), hD_t + P] &\in \mathcal{C}^\infty([0, T]; h^\infty \Psi^{\text{comp}}(X)). \end{aligned} \tag{A.6}$$

(These assumptions imply that  $\varphi_t(\text{WF}_h(A)) \subset \text{WF}(Q_j(t)) \subset \text{WF}(Q_{j+1}(t)) \subset \text{WF}(Q(t))$ .)

Let  $v(t) \stackrel{\text{def}}{=} e^{-it(P-iW)/h} Au$ ,  $\|u\|_{L^2} = 1$ . Our aim is to prove the following property:

$$(A.7) \quad w_j(t) \stackrel{\text{def}}{=} (I - Q_j(t))v(t) = \mathcal{O}(h^{j/2})_{L^2}, \quad \text{for } j = 0, \dots, N, \quad 0 \leq t \leq T.$$

Since  $A \in \Psi^{\text{comp}}$ , (A.2) shows that this property holds for  $j = 0$ . Let us now prove that, if true at the level  $j$ , it then holds at the level  $j + 1$ .

Noting that

$$(A.8) \quad w_{j+1} = (I - Q_{j+1})w_j + \mathcal{O}(h^\infty)_{\mathcal{C}^\infty},$$

we have

$$(hD_t + P - iW)w_{j+1} = (I - Q_{j+1}(t))(hD_t + P - iW)w_j - i[W, Q_{j+1}]w_j + \mathcal{O}(h^\infty)_{L^2}$$

Dividing by  $h/i$ , taking the inner product with  $w_{j+1}$ , taking real parts and integrating gives

$$\begin{aligned} (A.9) \quad & \|w_{j+1}(t)\|_{L^2}^2 - \|w_{j+1}(0)\|_{L^2}^2 + 2 \int_0^t \langle W w_{j+1}(s), w_{j+1}(s) \rangle ds = \\ & \frac{2}{h} \int_0^t \text{Re} \langle [W, Q_{j+1}(s)] w_j(s), w_{j+1}(s) \rangle ds + \mathcal{O}(h^\infty), \end{aligned}$$



Now,

$$(I - Q_{j+1}(s))[W, (I - Q_{j+1}(s))] = ihB_{j+1}(s) + h^2C_{j+1}(s),$$

$$B_{j+1}(s), C_{j+1}(s) \in \Psi^{\text{comp}}(X), \quad B_{j+1}(s) = B_{j+1}(s)^*.$$

Hence, using (A.8) and the induction hypothesis (A.7), the right hand side of (A.9) becomes

$$2h \int_0^t \text{Re} \langle C_{j+1}(s)w_j(s), w_j(s) \rangle ds + \mathcal{O}(h^\infty) = \mathcal{O}(h^{j+1}).$$

Returning to (A.9) and using the non-negativity of  $W$ , we see that

$$\|w_{j+1}(t)\|_{L^2}^2 \leq \|w_{j+1}(0)\|_{L^2}^2 + Ch^{j+1}.$$

Since

$$w_{j+1}(0) = (I - Q_{j+1})Au = \mathcal{O}(h^\infty)_{L^2},$$

we have established (A.7) with  $j$  replaced by  $j + 1$ .

The estimate (A.4) then follows from

$$(I - Q(t))v(t) = (I - Q(t))w_j(t) + \mathcal{O}_{L^2}(h^\infty),$$

the estimate (A.7) at the level  $j = N$ , and the fact that  $N$  could be taken arbitrary large.

To see (A.5) we note that if  $A \in \Psi^{\text{comp}}(X)$  then

$$\text{WF}_h(A) \subset w^{-1}([\epsilon, \infty) \implies \varphi_t(\text{WF}_h(A)) \subset w^{-1}([\epsilon/2, \infty) \quad \text{for } 0 \leq t \leq \delta.$$

Hence, by (A.4),

$$\text{WF}_h(v(t)) \subset w^{-1}([\epsilon/2, \infty), \quad v(t) \stackrel{\text{def}}{=} e^{-it(P-iW)/h}Au, \quad \|u\|_{L^2} = 1, \quad 0 \leq t \leq \delta.$$

This means that we can modify  $W$  into  $W_1$ , so that

$$\sigma(W_1)(x, \xi) \geq \langle \xi \rangle^k / C, \quad W_1 \geq c_0, \quad \text{for } 0 < h < h_0,$$

while we have

$$0 = (hD_t + P - iW)v(t) = (hD_t + P - iW_1)v(t) + \mathcal{O}(h^\infty)_{C^\infty} \quad \text{uniformly for } 0 \leq t \leq \delta.$$

Taking the imaginary part of the inner product of the above expression with  $v(t)$  gives

$$\frac{h}{2} \partial_t \|v(t)\|_{L^2}^2 = -\langle W_1 v(t), v(t) \rangle + \mathcal{O}(h^\infty) \leq -c_0 \|v(t)\|^2 + \mathcal{O}(h^\infty),$$

and hence

$$\|v(t)\|_{L^2}^2 = \mathcal{O}(h^\infty) \quad \text{uniformly for } \delta/2 \leq t \leq \delta.$$

This proves the first part of (A.5). The second part follows by taking a conjugate:  $A e^{-it(P-iW)/h} = (e^{-it(-P-iW)/h} A^*)^*$ , and all the arguments remain valid for  $P$  replaced by  $-P$ .  $\square$

The next lemma is needed in §7 and follows immediately from Lemma A.1:

**Proposition A.2.** *Suppose that  $A \in \Psi^{\text{comp}}(X)$  satisfies*

$$(A.10) \quad \text{WF}_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0)),$$

*for some  $\epsilon_0 > 0$  and that  $T$  is independent of  $h$ .*

*Then there exists  $B \in \Psi^{\text{comp}}(X)$  for which (A.10) holds with  $B$  in place of  $A$ , and*

$$(A.11) \quad e^{-it(P-iW)/h} A = B e^{-it(P-iW)/h} A + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}, \quad 0 \leq t \leq T.$$

*Proof.* Using again the operator  $Q(t)$  constructed in the proof of Lemma A.1, we take a compact set  $L$  containing  $\text{WF}_h(Q(t))$  for all  $0 \leq t \leq T$ . By taking  $\text{WF}_h(Q(0)) \subset p^{-1}((-\delta, \delta))$  (which is possible due the assumptions on  $A$ ) we see that we can assume  $L \subset p^{-1}((-\delta, \delta))$ . We can now choose  $B \in \Psi^{\text{comp}}(X)$  such that

$$\text{WF}_h(I - B) \cap L \cap w^{-1}([0, \epsilon_0/3]) = \emptyset, \quad \text{WF}_h(B) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_0/2]).$$

This implies that  $(I - B)Q(t) = C(t)$ , where  $\text{WF}_h(C(t)) \subset w^{-1}([\epsilon_0/3, \infty))$ , and hence, by (A.4) and (A.5),

$$(I - B)e^{-it(P-iW)/h} A = (C(t) + (I - B)(I - Q(t))e^{-it(P-iW)/h} A = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

proving (A.11).  $\square$

Finally we present a modification of [32, Lemma A.1]. The modification lies in slightly different assumptions on  $P$  and  $W$ , and the proof also corrects a mistake in the proof given in [32]. From now on we work under the extra assumption (1.10) on the CAP. We remark that in [32] we only needed Lemma A.1 and hence the assumption (1.10) was not required.

**Proposition A.3.** *Suppose that  $X$  is a compact manifold,  $P$  is a self-adjoint operator,  $P \in \Psi^m(X)$ ,  $W \in \Psi^k(X)$ ,  $W \geq 0$ , and that (1.9) and (1.10) hold. Then for any  $t$  independent of  $h$ , for  $A \in \Psi^{\text{comp}}(X)$  satisfying (A.10), we may write*

$$e^{itP/h} e^{-it(P-iW)/h} A = V_A(t) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

where

$$(A.12) \quad \begin{aligned} V_A(t) &\in \Psi_\delta^{\text{comp}}(X), \quad \text{WF}_h(V_A(t)) \subset \bigcap_{0 \leq s \leq t} (\varphi_{-s}(w^{-1}(0))) \cap \text{WF}_h(A), \\ \sigma(V_A(t)) &= \exp\left(-\frac{1}{h} \int_0^t \varphi_s^* W ds\right) \sigma(A). \end{aligned}$$

The class of operators  $\Psi_\delta^{\text{comp}}$  was introduced in §3.2.

The proof is based on the following lemma inspired by the pseudodifferential approach to constructing parametrices for parabolic equations presented in [28].

**Lemma A.4.** Suppose that  $t \mapsto p(t, z, h), p(t, \bullet, h) \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}; \mathbb{R})$ , is a family of functions satisfying

$$(A.13) \quad \begin{aligned} \partial_t^k \partial_z^\alpha p(t, z, h) &= \mathcal{O}_{k, \alpha}(1), \quad p \geq -Ch, \quad 0 < h < h_0, \\ |\partial_z^\alpha p(t, z, h)| &= \mathcal{O}_\alpha(p^{1-\delta}), \quad 0 < \delta < \frac{1}{2}. \end{aligned}$$

Then, for  $0 \leq s \leq t$  there exists  $E(t, s) \in \Psi_\delta(\mathbb{R}^n)$  such that

$$(h\partial_t + p^w(t, x, hD_x, h))E(t, s) = 0, \quad t \geq s \geq 0, \quad E(s, s) = I.$$

Moreover,  $E(t, s) = e^w(t, s, x, hD_x, h)$  where  $e(t, s) \in S_\delta(\mathbb{R}^{2n})$  has an explicit expansion given in (A.26) below.

*Proof.* Replacing  $p$  by  $p + (C+1)h$ , gives  $p \geq h$  and  $p(t, \bullet, h) \in (C+1)h + \mathcal{C}_c^\infty(\mathbb{R}_z^{2n})$ . The multiplicative factor  $e^{(C+1)(t-s)}$  in the evolution equation is irrelevant to our estimates.

For any  $N \geq 0$  we try to approximate the symbol  $e(t, s, x, \xi, h)$  by an expansion of the form

$$(A.14) \quad f_N(t, s, z, h) \stackrel{\text{def}}{=} \sum_{j=0}^N h^j e_j(t, s, z, h).$$

The symbol of the operator  $h\partial_t f_N^w + p^w f_N^w$  can be expanded using the standard notation  $a^w \circ b^w = (a \# b)^w$  and the product formula (see for instance [43, Theorem 4.12]):

$$\begin{aligned} & h\partial_t f_N(t, s) + [p(t) \# f_N(t, s)] \\ &= \sum_{j=0}^N h^j \left( h\partial_t e_j(t, s) + \sum_{k=0}^{N-j-1} \frac{1}{k!} \left( \frac{1}{2} i h \omega(D_z, D_w) \right)^k p(t, z) e_j(t, s, w)|_{z=w} + h^{N-j} r_{N,j} \right) \\ &= \sum_{j=0}^N h^j \left( (h\partial_t + p(t)) e_j(t, s) + \sum_{\ell=0}^{j-1} \frac{1}{(j-\ell)!} \left( \frac{1}{2} i \omega(D_z, D_w) \right)^{j-\ell} p(t, z) e_\ell(t, s, w)|_{z=w} \right) \\ & \quad + h^N r_N(t, s, z), \quad r_N(t, s, z) \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} r_{N,j}(t, s, z). \end{aligned}$$

The remainders satisfy the following bounds (see for instance [35, (3.12)]):

$$(A.15) \quad \sup_z |\partial_z^\alpha r_{N,j}(t, s, z)| \leq C_{\alpha, N, j} \sum_{\alpha_1 + \alpha_2 = \alpha} \sup_{z, w} \sup_{|\beta| \leq M, \beta \in \mathbb{N}^{4n}} \left| (h^{\frac{1}{2}} \partial_{z, w})^\beta (\sigma(D_z, D_w))^{N-j} \partial_z^{\alpha_1} p(z) \partial_w^{\alpha_2} e_j(w) \right|.$$

The standard strategy is now to iteratively construct the symbols  $e_j$  so that each term in the above expansion vanishes. The term  $j = 0$  simply reads  $(h\partial_t + p)e_0 = 0$ . From the

initial condition  $e_0(s, s) \equiv 1$ , it is solved by

$$(A.16) \quad e_0(t, s, z, h) = \exp \left( -\frac{1}{h} \int_s^t p(s', z, h) ds' \right).$$

For  $j \geq 1$ , the symbol  $e_j$  is obtained iteratively by solving

$$(A.17) \quad \begin{aligned} e_j(t, s, z) &\stackrel{\text{def}}{=} \frac{1}{h} \int_s^t e_0(t, s', z) q_j(s', s, z) ds', \quad e_j(t, s, \bullet) \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}), \\ q_j(t, s, z) &\stackrel{\text{def}}{=} - \sum_{\ell=0}^{j-1} \frac{1}{(j-\ell)!} \left( \frac{1}{2} i \omega(D_z, D_w) \right)^{j-\ell} p(t, z) e_\ell(t, s, w)|_{z=w} \in \mathcal{C}_c^\infty(\mathbb{R}_z^{2n}). \end{aligned}$$

This construction formally leads to an approximate solution:

$$(A.18) \quad h \partial_t f_N(t, s, z) + [p(t, \bullet) \# f_N(t, s, \bullet)](z) = h^N r_N(t, s, z).$$

To make the approximation effective, we now need to check that the sum (A.14) is indeed an expansion in power of  $h$ . We thus need to estimate the  $e_j$ 's and thereby the remainders  $r_{N,j}$ 's.

We will prove the following estimate by induction:

$$(A.19) \quad |\partial_z^\alpha e_j(t, s, z)| \leq C_{\alpha,j} h^{-2\delta j - \delta|\alpha|} \left( 1 + \left( \frac{1}{h} \int_s^t p(s', z) ds' \right)^{2j+|\alpha|} \right) e_0(t, s, z).$$

For that we first note that, as  $p \geq h$ , and  $|\partial^\alpha p| \leq C_\alpha p^{1-\delta}$ , we have

$$(A.20) \quad |\partial^\alpha p| \leq C_\alpha h^{-\delta} p.$$

Consequently, for  $j = 0$  we have

$$(A.21) \quad \begin{aligned} |\partial_z^\alpha e_0(t, s, z)| &\leq \sum_{\sum_{\ell=1}^k \alpha_\ell = \alpha} \prod_{\ell=1}^k \left( \frac{1}{h} \int_s^t |\partial^{\alpha_\ell} p(s', z)| ds' \right) e_0(t, s, z) \\ &\leq C_\alpha \sum_{\sum_{\ell=1}^k \alpha_\ell = \alpha} \prod_{\ell=1}^k \left( h^{-\delta} \frac{1}{h} \int_s^t p(s', z) ds' \right) e_0(t, s, z) \\ &\leq C'_\alpha h^{-\delta|\alpha|} \left( 1 + \left( \frac{1}{h} \int_s^t p(s', z) ds' \right)^{|\alpha|} \right) e_0(t, s, z), \end{aligned}$$

Here we used the fact that  $k \leq |\alpha|$  and that

$$A^k \leq c_\alpha (1 + A^{|\alpha|}), \quad A = \frac{1}{h} \int_s^t p(s', z) ds' \geq 0.$$

This gives (A.19) for  $j = 0$ .

To proceed with the induction we put

$$a_{j,\alpha}(t, s, z) \stackrel{\text{def}}{=} \partial_z^\alpha e_j(t, s, z) / e_0(t, s, z), \quad b_{j,\alpha}(t, s, z) \stackrel{\text{def}}{=} \partial_z^\alpha q_j(t, s, z) / e_0(t, s, z),$$

noting that, for some coefficients,  $c_\bullet$ ,

$$(A.22) \quad \begin{aligned} b_{j,\alpha}(t, s, z) &= \sum_{\ell=0}^{j-1} \sum_{\beta_1+\beta_2=\alpha} c_{\beta_1,\beta_2,\ell,j} \omega(D_z, D_w)^{j-\ell} \partial_z^{\beta_1} p(t, z) a_{\ell,\beta_2}(t, s, w)|_{z=w}, \\ a_{j,\alpha}(t, s, z) &= \frac{1}{h} \sum_{\beta_1+\beta_2=\alpha} c_{\beta_1,\beta_2,j} \int_s^t a_{0,\beta_1}(t, s', z) b_{j,\beta_2}(s', s) ds', \end{aligned}$$

where the last equality follows from  $e_0(t, s', z) e_0(s', s, z) = e_0(t, s, z)$ ,  $s \leq s' \leq t$ .

Our aim is to show

$$(A.23) \quad |b_{j,\alpha}(t, s, z)| \leq C_{\alpha,j} h^{-2\delta j - \delta|\alpha|} p(t, z) \left( 1 + \left( \frac{1}{h} \int_s^t p(s', z) ds' \right)^{2j+|\alpha|-1} \right),$$

and

$$(A.24) \quad |a_{j,\alpha}(t, s, z)| \leq C'_{\alpha,j} h^{-2\delta j - \delta|\alpha|} \left( 1 + \left( \frac{1}{h} \int_s^t p(s', z) ds' \right)^{2j+|\alpha|} \right),$$

assuming the statements are true for  $j$  replaced by smaller values.

We note that the case of  $j = 0$  has been shown in (A.21), and since  $b_{0,\alpha} \equiv 0$ .

The first estimate (A.23) follows immediately from the inductive hypothesis on  $a_{\ell,\alpha}$ ,  $0 \leq \ell \leq j-1$  and the estimates on  $p$  in (A.20). The second estimate (A.24) follows from (A.21), (A.23) and the obvious fact that  $\int_{s_1}^{s_2} p(s') ds' \leq \int_s^t p(s') ds'$ ,  $s \leq s_1 \leq s_2 \leq t$ .

We note that (A.19) and the definition of  $e_0$  given in (A.16) imply that

$$\partial_z^\alpha e_j(t, s, z) = \mathcal{O}(h^{-\delta|\alpha| - 2\delta j}), \quad j \geq 0.$$

so from (A.14) we see that the symbol  $f_N(t, s) \in S_\delta(\mathbb{R}^{2n})$ .

The bounds (A.15) then show that the remainders satisfy

$$|\partial^\alpha r_N(t, s, z)| \leq C_{N,\alpha} h^{-2\delta N - \delta|\alpha|}.$$

Going back to (A.18) we get the expression

$$(A.25) \quad E(t, s) = f_N^w(t, s, x, hD_x) + h^{N-1} \int_s^t E(t, s') r_N^w(s', s, x, hD_x).$$

(We note that, since  $p^w(t, x, hD_x) \geq -Ch$  by the sharp Gårding inequality [43, Theorem 4.32], and since  $p^w$  is bounded on  $L^2$ , the operator  $E(t, s)$  exists and is bounded on  $L^2$ , uniformly in  $h$ .) Since operators in  $\Psi_\delta$  are uniformly bounded on  $L^2$  [43, Theorem 4.23], it follows that

$$E(t, s) = f_N^w(t, s, x, hD_x) + \mathcal{O}(h^{(1-2\delta)N})_{L^2 \rightarrow L^2}.$$

To show that  $E(t, s) - e_0^w(s, t, x, hD_x) \in \Psi_\delta^{\text{comp}}(\mathbb{R}^n)$ , we use (A.25) and Beals's lemma in the form given in [35, Lemma 3.5,  $\tilde{h} = 1$ ]:  $\ell_j$  are linear functions on  $\mathbb{R}^{2n}$ ,  $\ell_j^w = \ell_j^w(x, hD)$ ,

then

$$\begin{aligned} \text{ad}_{\ell_1^w} \cdots \text{ad}_{\ell_J^w} E(s, t) &= \\ \text{ad}_{\ell_1^w} \cdots \text{ad}_{\ell_J^w} f_N^w(s, t, x, hD_x) &+ h^{N-1} \int_s^t \text{ad}_{\ell_1^w} \cdots \text{ad}_{\ell_J^w} (E(s, s') r_N^w(s', s, x, hD_x)) ds' \\ &= \mathcal{O}(h^{(1-2\delta)J})_{L^2 \rightarrow L^2} + \mathcal{O}(h^{(1-2\delta)N})_{L^2 \rightarrow L^2} = \mathcal{O}(h^{(1-2\delta)J})_{L^2 \rightarrow L^2}, \end{aligned}$$

if  $N$  is large enough. Here we used the fact that  $f_N, r_N \in S_\delta$  and that  $\text{ad}_{\ell_1^w} \cdots \text{ad}_{\ell_J^w} E(s, t) = \mathcal{O}(1)_{L^2 \rightarrow L^2}$ , which follows from considering the evolutions equation for the operator on the left hand side.

In conclusion we have shown that  $E(t, s) = e^w(t, s, x, hD_x)$ , where  $e \in S_\delta(\mathbb{R}^n)$  admits the expansion

$$(A.26) \quad e(t, s, z, h) \sim \sum_{j \geq 0} h^j e_j(t, s, z, h), \quad e_j(t, s) \in h^{-2\delta j} S_\delta^{\text{comp}}(\mathbb{R}^{2n}), \quad j \geq 1,$$

with  $e_0$  given by (A.16). □

*Proof of Proposition A.3.* We first observe that Lemma A.1 (applied both to propagators for  $P - iW$  and for  $P$ ) shows that for  $B \in \Psi^{\text{comp}}(X)$  satisfying  $\text{WF}_h(I - B) \cap \text{WF}_h(A) = \emptyset$ ,

$$e^{itP/h} e^{-it(P-iW)/h} A = B e^{itP/h} e^{-it(P-iW)/h} A + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

We can choose  $B = B^*$ . Since

$$\begin{aligned} h\partial_t (B e^{itP/h} e^{-it(P-iW)/h} A) &= -B e^{itP/h} W e^{-itP/h} e^{itP/h} e^{-it(P-iW)/h} A \\ &= - (B e^{itP/h} W e^{-itP/h} B) (B e^{itP/h} e^{-it(P-iW)/h} A) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}, \end{aligned}$$

it follows that

$$(A.27) \quad B e^{itP/h} e^{-it(P-iW)/h} A = V^B(t) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

where

$$(A.28) \quad h\partial_t V^B(t) = -W_B(t) V^B(t), \quad W_B(t) \stackrel{\text{def}}{=} B e^{itP/h} W e^{-itP/h} B.$$

We note that  $W_B(t) \in \Psi^{\text{comp}}(X)$ ,  $\text{WF}_h(W_B(t)) \subset \text{WF}_h(B)$ , and that  $W_B(t) \geq 0$ . Hence  $V^B(t) = \mathcal{O}(1)_{L^2 \rightarrow L^2}$  and (A.27) follows from Duhamel's formula.

By decomposing  $A$  as a sum of operators, we can assume that  $\text{WF}_h(A)$  is supported in a neighbourhood of a fiber of a point in  $X$ . Hence, by choosing  $B$  with a sufficiently small wave front set, we only need to prove that  $V^B(t) \in \Psi_\delta$  for  $X = \mathbb{R}^n$ ; that follows from Lemma A.4, since the symbol of  $W_B(t)$  satisfies the assumptions (A.13). The second and third properties in (A.12) follows from (A.5) and (A.26). □

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